IC8451 – control systems

Unit-3 Frequency Response Analysis

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① Sketch the Bode plot for the following transfer function and determine phase margin and gain margin.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$$

Soln:

$$s^2+16s+100 = s^2+2\xi\omega_n s+\omega_n^2$$

$$\omega_n^2 = 100, \quad \omega_n = 10 = \omega_{c2} \text{ rad/sec}$$

$$\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec.}$$

Bode form $G(s) = \dfrac{75(1+0.2s)}{s\,100\left(1+\frac{s^2}{100}+\frac{16s}{100}\right)} = \dfrac{0.75(1+0.2s)}{s(1+0.01s^2+0.16s)}$

put $s=j\omega$, $G(j\omega) = \dfrac{0.75}{j\omega}\dfrac{(1+0.2j\omega)}{(1-0.01\omega^2+j0.16\omega)}$

$\omega_l << \omega_{c1} = 0.5$ rad/sec ; $\omega_h >> \omega_{c2} = 20$ rad/sec

Basic factor	Corner frequency (rad/sec)	Slope in dB/decade	Change in slope in dB/decade
$\frac{0.75}{j\omega}$	–	−20	
$1+j0.2\omega$	$\omega_{c1}=5$	20	0
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	$\omega_{c2}=10$	−20	−20

add

add

at $\omega = \omega_l = 0.5$ rad/sec,

$$A_1 = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\left|\frac{0.75}{\omega}\right| = 20\log\left|\frac{0.75}{0.5}\right| = +3.5\,dB$$

at $\omega = \omega_{c1} = 5$ rad/sec

$$A_2 = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\left|\frac{0.75}{5}\right| = -16.5\,dB$$

at $\omega = \omega_{c2} = 10$ rad/sec

$$A_3 = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_2$$

$$= \left[0 \times \log\frac{10}{5}\right] - 16.5 = -16.5\,dB$$

at $\omega = \omega_h = 20$ rad/sec

$$A_4 = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_3$$

$$= \left[-20 \times \log\frac{20}{10}\right] - 16.5 = -28.5\,dB$$

Magnitude plot table:

ω (rad/sec)	0.5	5	10	20
$\lvert A\rvert$ in dB	+3.5	-16.5	-16.5	-28.5

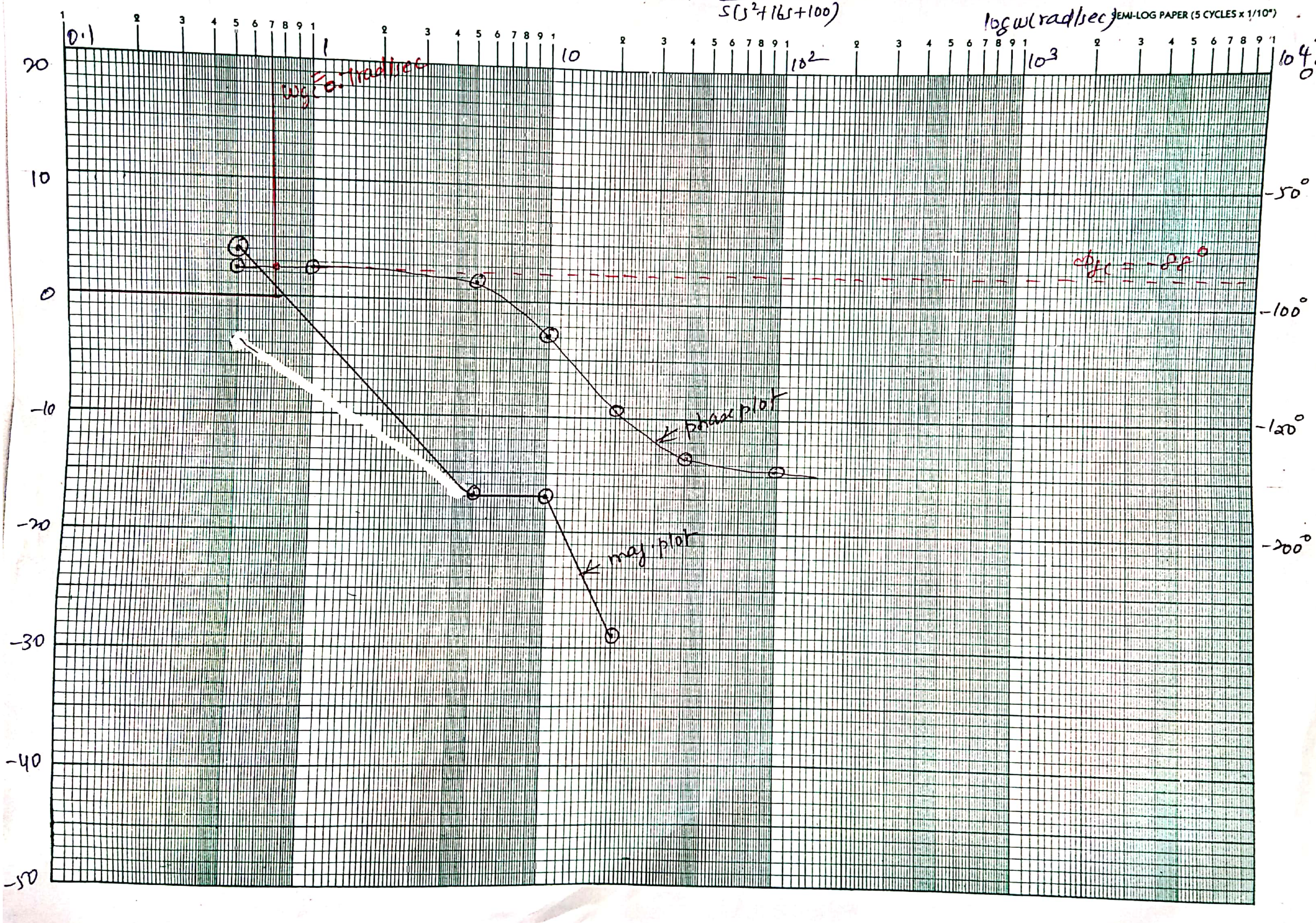
phase angle expression $\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}$ for $\omega < \omega_n$

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \left(\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} + 180^\circ\right) \text{ for } \omega > \omega_n$$

phase angle plot table

ω (rad/sec)	0.5	1	5	10	20	50	100
ϕ (deg)	-88°	-88°	-92°	-116°	-148°	-168°	-174°

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$$



Determination of Gain margin and phase margin:-

$$GM = -20\log |G(j\omega) \text{ at } \omega = \omega_{pc}|$$

The phase plot crosses -180° only at infinity. The $|G(j\omega)|$ at infinity is $-\infty$ dB.

Hence the gain margin GM is ∞

$$PM = 180^\circ + \angle G(j\omega) \text{ at } \omega = \omega_{gc}$$

from the graph, $\omega_{gc} = 0.7$ rad/sec,

at this gain cross over frequency, the phase angle ϕ_{gc} is -88°

$$PM = 180^\circ - 88^\circ = 92^\circ$$

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(2) Derive an expression for all the frequency domain specifications such as Resonant peak (M_r), Resonant frequency (ω_r), Bandwidth (ω_b) Gain margin (K_g) and phase margin (γ).

Resonant peak (M_r):

Consider the closed loop T.F of 2nd order system

$$\frac{C(s)}{R(s)} = M(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The sinusoidal T.F $M(j\omega)$ is obtained by putting $s = j\omega$

$$\therefore M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{-\omega^2 + j2\xi\omega_n\omega + \omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2\left[\frac{-\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n} + 1\right]} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\xi\frac{\omega}{\omega_n}}$$

Let Normalized frequency $u = \frac{\omega}{\omega_n}$

$$\therefore M(j\omega) = \frac{1}{(1-u^2) + j2\xi u}$$

Let $M \rightarrow$ magnitude of C.L.T.F

$\alpha \rightarrow$ phase of C.L.T.F

$$M = |M(j\omega)| = \left[\frac{1}{(1-u^2)^2 + (2\xi u)^2}\right]^{1/2} = \left[(1-u^2)^2 + 4\xi^2 u^2\right]^{-1/2} \quad (1)$$

$$\alpha = \angle M(j\omega) = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)$$

The resonant peak is the maximum value of M. The condition for maximum value of M can be obtained by differentiating the equation of M with respect to u and letting $\frac{dM}{du} = 0$ when $u = u_r$

where $u_r = \frac{\omega_r}{\omega_n}$ = Normalized resonant frequency

On differentiating (1) equation w.r.t 'u'

$$\frac{dM}{du} = \frac{d}{du}\left[(1-u^2)^2 + 4\xi^2 u^2\right]^{-1/2}$$

$$= -\frac{1}{2}\left((1-u^2)^2 + 4\xi^2 u^2\right)^{-3/2}\left[2(1-u^2)(-2u) + 8\xi^2 u\right]$$

$$= \frac{-\left[-4u(1-u^2) + 8\xi^2 u\right]}{2\left[(1-u^2)^2 + 4\xi^2 u^2\right]^{3/2}} = \frac{4u(1-u)^2 - 8\xi^2 u}{2\left[(1-u^2)^2 + 4\xi^2 u^2\right]^{3/2}}$$

Replace u by u_r and equating to zero

$$\frac{4u_r(1-u_r^2) - 8\xi^2 u_r}{2\left[(1-u_r^2)^2 + 4\xi^2 u_r^2\right]^{3/2}} = 0$$

$$4u_r(1-u_r^2) - 8\xi^2 u_r = 0 \Rightarrow 4u_r - 4u_r^3 - 8\xi^2 u_r = 0$$

$$4u_r^3 = 4u_r - 8\xi^2 u_r$$

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$$u_r^2 = 1-2\xi^2$$

$$u_r = \sqrt{1-2\xi^2}$$

$\therefore$ The resonant peak occurs when $u_r = \sqrt{1-2\xi^2}$

$$\therefore M_r = \frac{1}{\left[(1-u_r^2)^2+4\xi^2u_r^2\right]^{1/2}} = \frac{1}{\left[(1-u_r^2)^2+4\xi^2u_r^2\right]^{1/2}}$$

$$= \frac{1}{\left[\left[1-(1-2\xi^2)\right]^2+4\xi^2(1-2\xi^2)\right]^{1/2}} = \frac{1}{\left[4\xi^4+4\xi^2-8\xi^4\right]^{1/2}}$$

$$= \frac{1}{(4\xi^2-4\xi^4)^{1/2}} = \frac{1}{\left[4\xi^2(1-\xi^2)\right]^{1/2}} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$\therefore$ Resonant peak $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

Resonant frequency (ω_r)

Normalized resonant frequency $u_r = \dfrac{\omega_r}{\omega_n} = \sqrt{1-2\xi^2}$

The resonant frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$

Bandwidth (ω_b):

Let normalized bandwidth $u_b = \dfrac{\omega_b}{\omega_n}$

When $u = u_b$ the magnitude M of the closed loop system is $\dfrac{1}{\sqrt{2}}$ or $-3\,dB$

In eqn ① put $u = u_b$ and equate to $1/\sqrt{2}$

$$\therefore M = \frac{1}{\left[(1-u_b^2)^2+4\xi^2u_b^2\right]^{1/2}} = 1/\sqrt{2}$$

Squaring on both sides and cross multiplying we can get

$$(1-u_b^2)^2+4\xi^2 u_b^2=2$$

$$1+u_b^4-2u_b^2+4\xi^2u_b^2=2$$

$$u_b^4-2u_b^2(1-\xi^2)-1=0$$

Let $x=u_b^2$

$$\therefore\quad x^2-2(1-2\xi^2)x-1=0$$

$$\therefore\quad x=\frac{2(1-2\xi^2)\pm\sqrt{4(1-2\xi^2)^2+4}}{2}$$

$$=\frac{2(1-2\xi^2)\pm 2\sqrt{(1+4\xi^4-4\xi^2)+1}}{2}$$

Let us take only the +ve sign $x=1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}$

but $u_b=\sqrt{x}$ $\therefore$ $u_b=\sqrt{x}=\left[1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}\right]^{1/2}$

Also $u_b=\dfrac{\omega_b}{\omega_n}$

Bandwidth $\omega_b=\omega_n u_b=\omega_n\left[(1-2\xi^2)+\sqrt{2-4\xi^2+4\xi^4}\right]^{1/2}$

Gain margin (K_g)

The gain margin of 2nd order system is infinite.

Phase margin (γ)

The open loop T.F of 2nd order system $G(s)=\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

The sinusoidal T.F $G(j\omega)$ is obtained by letting $s=j\omega$

$$G(j\omega)=\frac{\omega_n^2}{j\omega(j\omega+2\xi\omega_n)}=\frac{\omega_n^2}{\omega_n\left(\frac{j\omega}{\omega_n}\right)\omega_n\left(2\xi+j\frac{\omega}{\omega_n}\right)}=\frac{1}{j\frac{\omega}{\omega_n}\left(2\xi+j\frac{\omega}{\omega_n}\right)}$$

Let normalized frequency $u = \omega/\omega_n$

Substituting $u = \omega/\omega_n$

$$G(j\omega) = \frac{1}{ju(2\xi + ju)}$$

$$M = \text{magnitude of } G(j\omega) = |G(j\omega)| = \frac{1}{u\sqrt{4\xi^2 + u^2}} = \frac{1}{\sqrt{u^4 + 4\xi^2 u^2}} \quad -(1)$$

$$\alpha = \text{phase of } G(j\omega) = -90^\circ - \tan^{-1}\left(\frac{u}{2\xi}\right)$$

At the gain cross over frequency ω_{gc}, the magnitude of $G(j\omega)$ is unity.

Let normalized gain cross over frequency $u_{gc} = \omega_{gc}/\omega_n$

Substitute u by u_{gc} and equating to unity of (1) equation

$$\therefore \text{ at } u = u_{gc}, \quad |G(j\omega)| = \frac{1}{\sqrt{u_{gc}^4 + 4\xi^2 u_{gc}^2}} = 1$$

$$u_{gc}^4 + 4\xi^2 u_{gc}^2 = 1 \Rightarrow u_{gc}^4 + 4u_{gc}^2\xi^2 - 1 = 0$$

Let $x = u_{gc}^2$, $\therefore \; x^2 + 4\xi^2 x - 1 = 0$

$$\therefore x = \frac{-4\xi^2 \pm \sqrt{16\xi^4 + 4}}{2} = -2\xi^2 \pm \sqrt{4\xi^4 + 1}$$

Let us take only the +ve sign $\therefore x = -2\xi^2 + \sqrt{4\xi^4 + 1}$

but $u_{gc} = \sqrt{x}$, $\therefore u_{gc} = \left[-2\xi^2 + \sqrt{4\xi^4 + 1}\right]^{1/2}$

The phase margin $\gamma = 180^\circ + \angle G(j\omega)$ at $\omega = \omega_{gc}$, $u = u_{gc}$

$$\gamma = 180^\circ + \left(-90^\circ - \tan^{-1}\frac{u_{gc}}{2\xi}\right) = 90^\circ - \tan^{-1}\left[\frac{\left[-2\xi^2 + \sqrt{4\xi^4 + 1}\right]^{1/2}}{2\xi}\right]$$

③ Explain how the closed loop frequency response from open loop frequency response is determined? explain.

Two graphical methods are available to determine the closed loop frequency response from open loop frequency response. They are

1. Constant M and N circles 2. Nichols chart.

Constant M and N circles:

The magnitude of C.L.T.F with unity feedback can be shown to be in the form of circle for every value of M. These circles are called M-circles.

If the phase of C.L.T.F with unity feedback is α then it can be shown that $\tan\alpha$ will be in the form of circle for every value of α. these circles are called N circles.

The M and N circles are used to find the C.L. frequency response graphically from the O.L frequency response $G(j\omega)$ without calculating the magnitude and phase of the C.L.T.F at each frequency.

The M and N circles are available as standard chart. The chart consists of M and N circles superimposed on ordinary graph sheet. Using ordinary graph the locus of $G(j\omega)$ is sketched. The locus of $G(j\omega)$ will cut the M circles and N circles at various points. The intersection of $G(j\omega)$ locus with M and N circles gives the magnitude and phase of the closed loop system at frequencies corresponding to the cutting point of $G(j\omega)$.

The M and α for various values of ω are tabulated. The magnitude and phase response of closed loop system are sketched on semilog graph sheet by taking ω on the logarithmic scale on x axis.

The closed loop frequency response has 2 plots. They are

M V_s ω and α V_s ω.

M. circles:

Consider the C.L.T.F of unity feedback system, $M(s) = \frac{G(s)}{1+G(s)}$

put $s = j\omega$, $\therefore M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)}$

Let $G(j\omega) = x + jy$

where $x \to$ Real part of $G(j\omega)$
$y \to$ Imaginary part of $G(j\omega)$

$$\therefore M(j\omega) = \frac{x+jy}{1+x+jy} = \frac{\sqrt{x^2+y^2} \ \tan^{-1}(y/x)}{\sqrt{(1+x)^2+y^2} \ \tan^{-1}\left(\frac{y}{1+x}\right)}$$

Let $M \to$ magnitude of $M(j\omega)$

$$M = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}$$

Squaring on both the sides,

$$M^2 = \frac{x^2+y^2}{(1+x)^2+y^2} \Rightarrow M^2\left[(1+x)^2+y^2\right] = x^2+y^2$$

$$M^2\left[1+x^2+2x+y^2\right] = x^2+y^2$$

$$x^2(M^2-1)+M^2 2x + M^2 + y^2(M^2-1) = 0$$

when $M=1$, the equation represents a straight line

when $M=1$, $2x+1=0 \Rightarrow x = -\frac{1}{2}$ [passing through $x=-\frac{1}{2}$ and $y=0$]

when $M \neq 1$ the equation represents a family of circles

$$x^2(M^2-1)+M^2 2x+M^2+y^2(M^2-1)=0$$

$\div \ M^2-1$

$$x^2 + \frac{M^2}{M^2-1} 2x + \frac{M^2}{M^2-1} + y^2 = 0$$

Add $M^2/(M^2-1)^2$ on both the sides

$$x^2 + \frac{M^2}{M^2-1} 2x + \frac{M^2}{M^2-1} + \frac{M^2}{(M^2-1)^2} + y^2 = \frac{M^2}{(M^2-1)^2}$$

$$x^2 + \frac{M^2}{M^2-1} 2x + \frac{M^2(M^2-1)+M^2}{(M^2-1)^2} + y^2 = \frac{M^2}{(M^2-1)^2}$$

$$x^2 + \frac{M^2}{M^2-1} 2x + \frac{M^4}{(M^2-1)^2} + y^2 = \frac{M^2}{(M^2-1)^2}$$

$$\left[x + \frac{M^2}{(M^2-1)}\right]^2 + y^2 = \frac{M^2}{(M^2-1)^2} \quad -①$$

The equation of circle with centre at (x_1, y_1) and radius r

$$(x-x_1)^2 + (y-y_1)^2 = r^2 \quad -②$$

Comparing ① and ② equations, centre $\left(-\frac{M^2}{M^2-1}, 0\right)$ radius $r = \frac{M}{M^2-1}$

when $M=0$

centre $= (x_1, y_1)$

$$x_1 = -\frac{M^2}{M^2-1} = 0$$

$$y_1 = 0$$

Radius $r = \frac{M}{M^2-1} = 0$

when $M = \infty$

centre $= (x_1, y_1)$

$$x_1 = -\frac{M^2}{M^2-1} \simeq \frac{-M^2}{M^2} \simeq -1$$

$$y_1 = 0$$

Radius $r = \frac{M}{M^2-1} \simeq \frac{M}{M^2} = \frac{1}{M} = \frac{1}{\infty} = 0$

Hence when $M=0$ the magnitude circle becomes a point at $(0,0)$

Hence when $M=\infty$ the magnitude circle becomes a point at $(-1,0)$

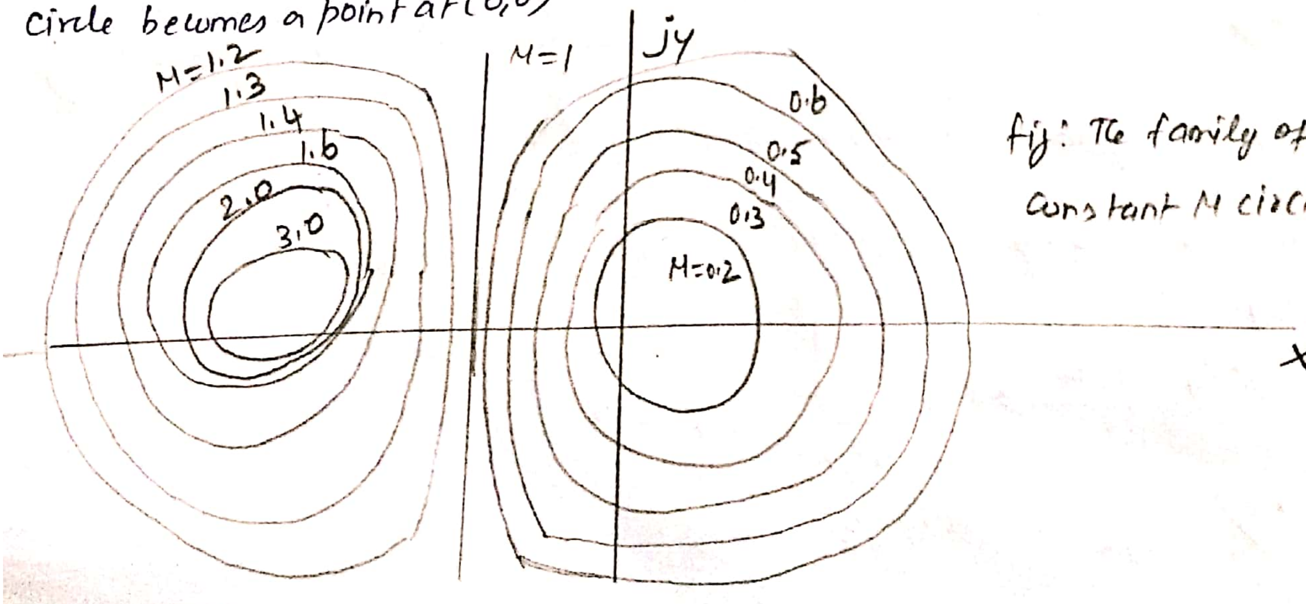


fig: The family of constant M circles.

N circles;

Consider the C.L.T.F of unity feedback system,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = M(s)$$

put $s=j\omega$, $M(j\omega) = \dfrac{G(j\omega)}{1+G(j\omega)}$

Let $G(j\omega) = x+jy$, where $x \rightarrow$ Real part of $G(j\omega)$

$y \rightarrow$ Imaginary part of $G(j\omega)$

$$\therefore M(j\omega) = \frac{x+jy}{1+x+jy} = \frac{\sqrt{x^2+y^2}\ \tan^{-1} y/x}{\sqrt{(1+x)^2+y^2}\ \tan^{-1} \frac{y}{1+x}}$$

Let α = phase of $M(j\omega)$

$$\alpha = \tan^{-1} y/x - \tan^{-1}\frac{y}{1+x}$$

Let $N = \tan\alpha$,

$$N = \tan\left[\tan^{-1}\frac{y}{x} - \tan^{-1}\frac{y}{1+x}\right]$$

W.K.T $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$

$$N = \frac{\tan\tan^{-1} y/x - \tan\tan^{-1}\frac{y}{1+x}}{1+\tan\tan^{-1}\frac{y}{x} \cdot \tan\tan^{-1}\frac{y}{1+x}} = \frac{y/x - \frac{y}{1+x}}{1+ y/x \cdot \frac{y}{1+x}}$$

$$N = \frac{\frac{y(1+x)-xy}{x(1+x)}}{1+ y/x \cdot \frac{y}{1+x}} = \frac{y+xy-xy}{x+x^2+y^2} = \frac{y}{x+x^2+y^2}$$

$$x+x^2+y^2 = y/N$$

$$x+x^2+y^2-y/N = 0$$

add the term $\frac{1}{4}+\left(\frac{1}{2N}\right)^2$ on both the sides

$$x+x^2+y^2-y/N+\frac{1}{4}+\left(\frac{1}{2N}\right)^2 = \frac{1}{4}+\left(\frac{1}{2N}\right)^2$$

$$\left(x^2+\tfrac{1}{4}+x\right)+\left(y^2+\left(\tfrac{1}{2N}\right)^2-\tfrac{y}{N}\right)=\tfrac{1}{4}+\left(\tfrac{1}{2N}\right)^2$$

$$\left(x+\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2} \quad -①$$

The equation of circle with centre at (x_1, y_1) and radius r is

$$(x-x_1)^2+(y-y_1)^2=r^2 \quad -②$$

on comparing ① & ② equations, centre at $\left(-\frac{1}{2}, \frac{1}{2N}\right)$ with radius $\sqrt{\frac{1}{4}+\left(\frac{1}{2N}\right)^2}$

for any value of N the equation of N circles is satisfied at two points $(0,0)$ and $(-1,0)$. Hence the N circles passes through these 2 points for all values of α. ($N=\tan\alpha$)

Consider the equation of N circle, when $x=0$ and $y=0$

$$\left(x+\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\left(\tfrac{1}{2}\right)^2+\left(-\tfrac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

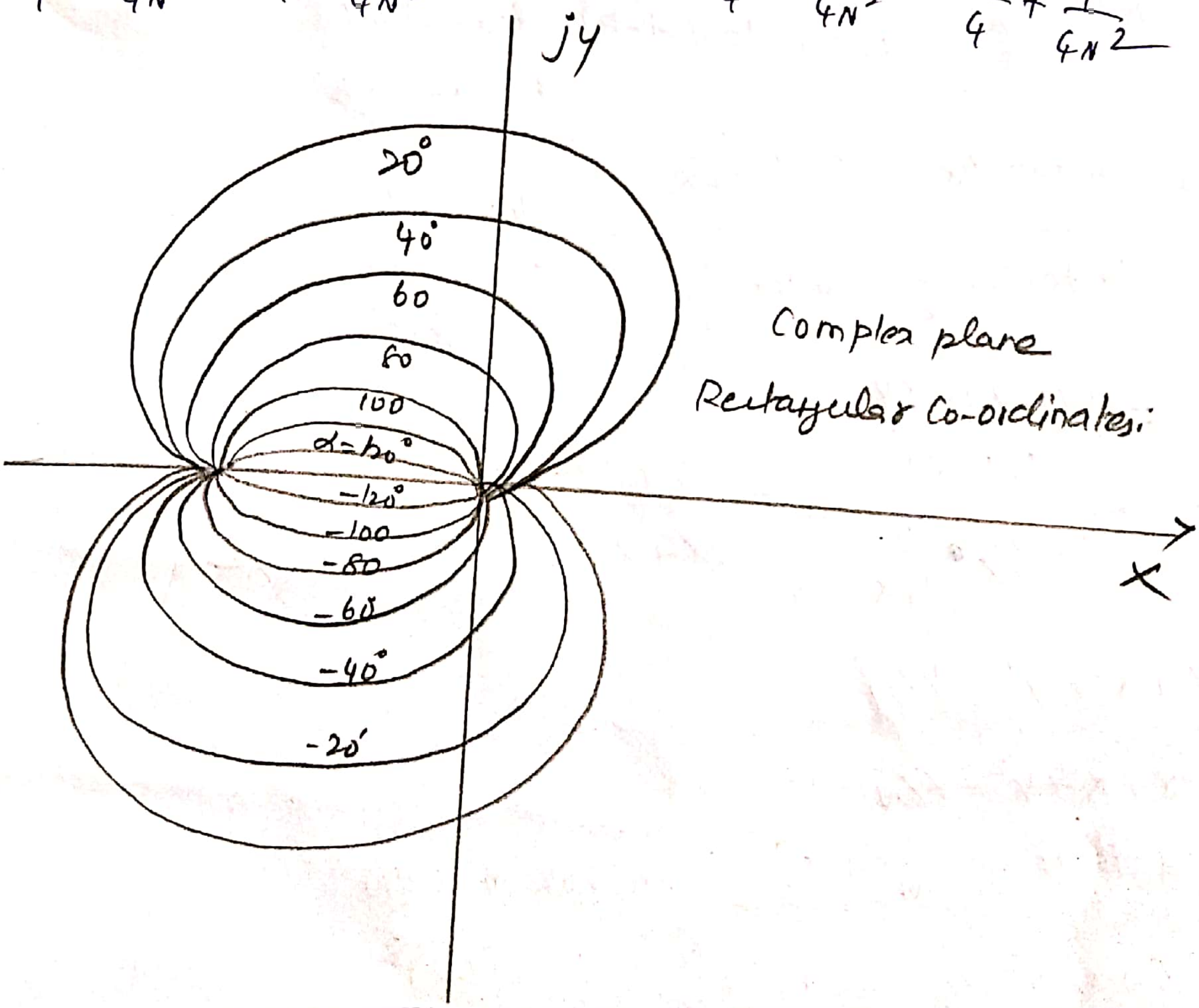
$$\frac{1}{4}+\frac{1}{4N^2}=\frac{1}{4}+\frac{1}{4N^2}$$

Consider the equation of N circle when $x=-1$ and $y=0$

$$\left(x+\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\left(-1+\tfrac{1}{2}\right)^2+\left(-\tfrac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\frac{1}{4}+\frac{1}{4N^2}=\frac{1}{4}+\frac{1}{4N^2}$$



Complex plane
Rectangular Co-ordinates.

④ The open loop T.F of a unity feedback system is given by $G(s) = \frac{1}{s(1+s)^2}$ Sketch the polar plot and determine the gain margin and phase margin.

Soln:-

$$G(s) = \frac{1}{s(1+s)^2}$$

put $s = j\omega$, $G(j\omega) = \frac{1}{j\omega(1+j\omega)^2} = \frac{1}{j\omega(1+j\omega)(1+j\omega)}$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+\omega^2}} = \frac{1}{\omega\left(\sqrt{1+\omega^2}\right)^2} = \frac{1}{\omega(1+\omega^2)} = \frac{1}{\omega+\omega^3}$$

$$\angle G(j\omega) = \phi = -90^\circ - 2\tan^{-1}\omega$$

ω(rad/sec)	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$\lvert G(j\omega)\rvert$	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4
$\angle G(j\omega)$	-134°	-143°	-151°	-159°	-167°	-174°	-180°	-185°

Determination of GM and PM:

$$GM = K_g = \frac{1}{|G(j\omega) \text{ at } \omega = \omega_{pc}|} = \frac{1}{0.5} = 2$$

$$PM = \gamma = 180^\circ + \angle G(j\omega) \text{ at } \omega = \omega_{gc} = 180^\circ - 159^\circ = 21^\circ$$

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⑤ Discuss the correlation between frequency domain and time domain specifications.

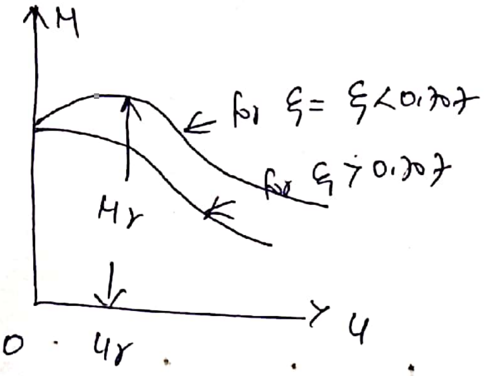
The correlation between time and frequency response has an explicit form only for first order and 2nd order systems.

Consider the magnitude and phase of a C.L. Second order system as a function of normalised frequency.

Magnitude of closed loop system, $M = |M(j\omega)| = \dfrac{1}{\left[(1-u^2)^2 + (2\xi u)^2\right]^{1/2}}$

Phase of closed loop system $\alpha = \angle M(j\omega) = -\tan^{-1}\dfrac{2\xi u}{1-u^2}$

fig: magnitude M as a function of u



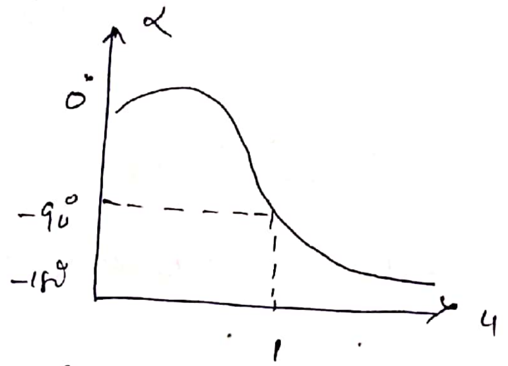


fig: phase α as a function of u

The frequency at which M has a peak value is known as the resonant frequency. The peak value of the magnitude is the resonant peak M_r. At this frequency the slope of the magnitude curve is zero. The frequency corresponding to M_r is u_r which is the normalised resonant frequency.

Resonant peak $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

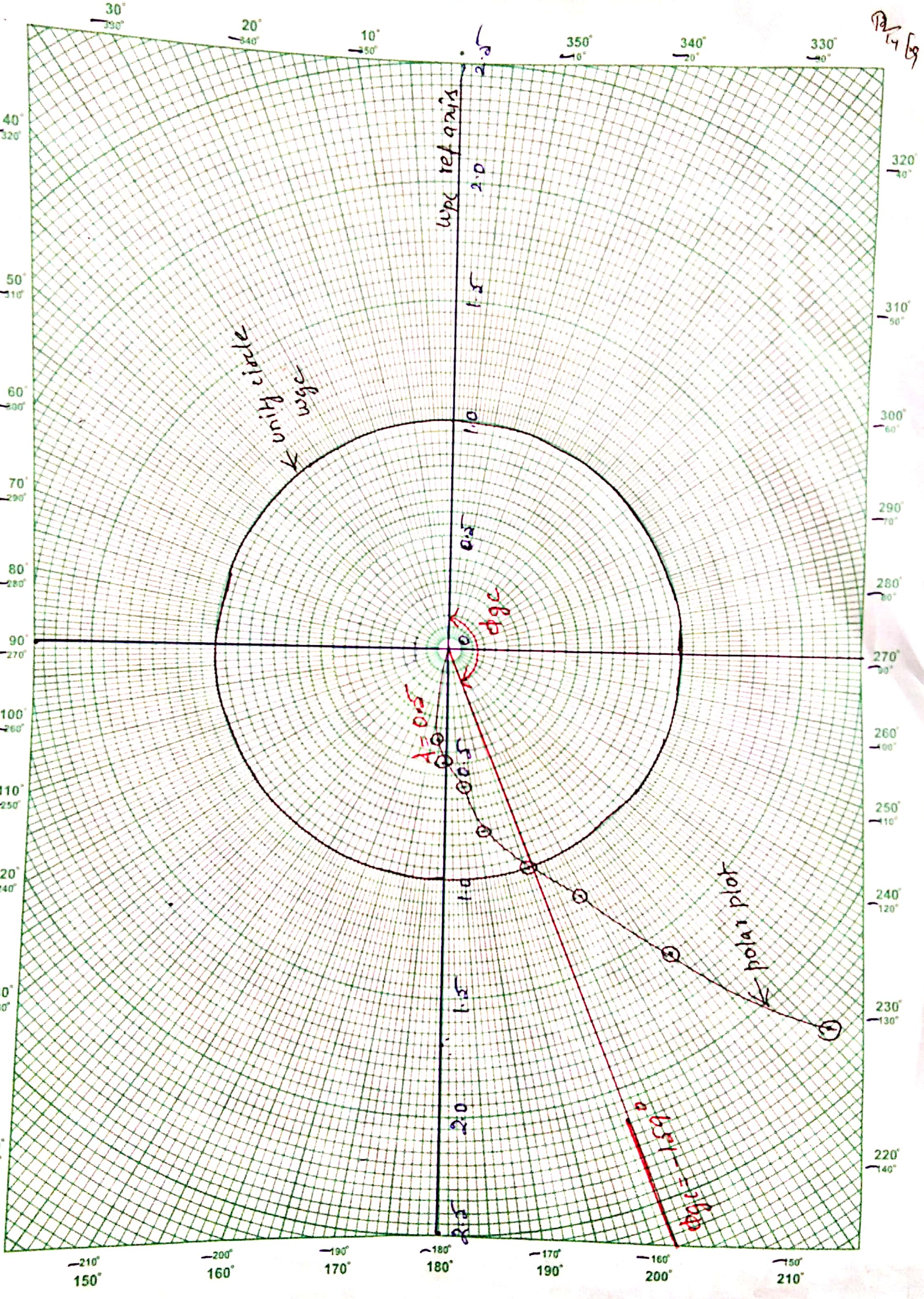
Resonant frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$

when $\xi = 0$, $\omega_r = \omega_n\sqrt{1-2\xi^2} = \omega_n$

when $\xi = 0$, $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

It is clear that as ξ tends to zero, ω_r approaches ω_n and M_r approaches infinity.

$$G(s) = \frac{1}{s(1+s)^2}$$



when $1-2\xi^2 = 0$, $\omega_r = 0$ which means there is no resonant peak at this condition.

let $1-2\xi^2 = 0$, $\xi^2 = 1/2 \Rightarrow \xi = \frac{1}{\sqrt{2}}$

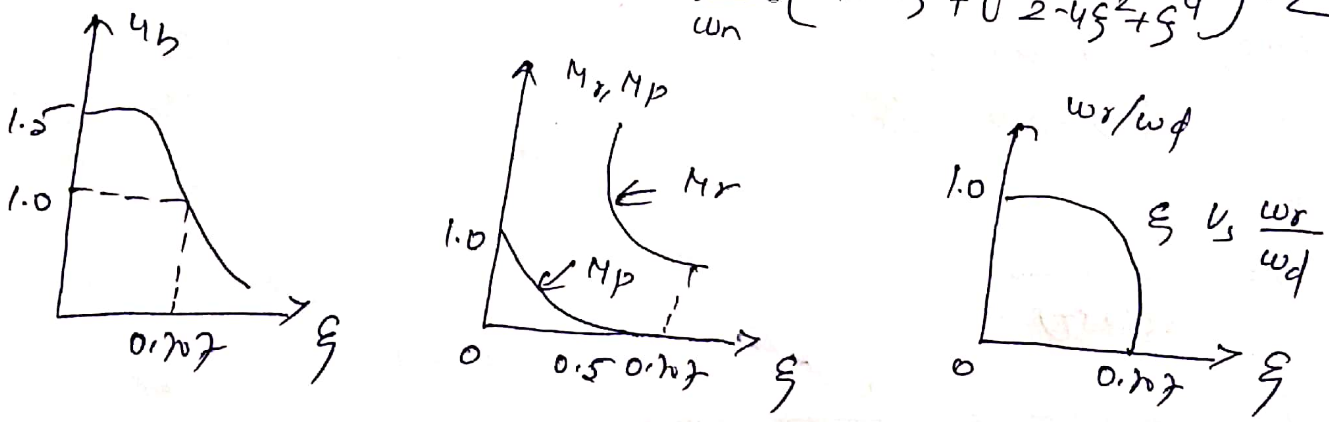
for $0 < \xi \leq 1/\sqrt{2}$ the resonant frequency always has a value $< \omega_n$ and Resonant peak M_r has a value greater than one.

for $\xi > 1/\sqrt{2}$ the condition $\frac{dM}{du} = 0$, will not be satisfied for any real value of ω.

Hence when $\xi > 1/\sqrt{2}$ the magnitude M decreases monotonically from $M=1$ at $u=0$ with Increasing u. It follows that for $\xi > 1/\sqrt{2}$ there is no resonant peak and the greatest value of M equals one.

The frequency at which M has a value of $\frac{1}{\sqrt{2}}$ is called as cut-off frequency ω_c. The signal frequencies above cut-off are attenuated on passing through a system.

The normalized bandwidth $u_b = \frac{\omega_b}{\omega_n} = \left[1-2\xi^2 + \sqrt{2-4\xi^2+\xi^4}\right]^{1/2}$



damped frequency $\omega_d = \omega_n\sqrt{1-\xi^2}$

peak overshoot $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$

(6) Draw the typical sketches of polar plot for the following

O.L.T.F $G(s) = \frac{1}{1+sT}$, $G(s) = \frac{1}{s(1+sT)}$

$G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$ $G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_2)}$

$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$ $G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$

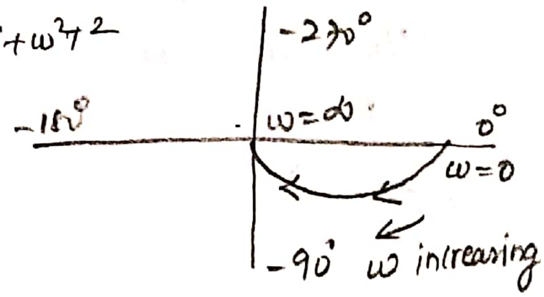
$G(s) = 1+sT$ $G(s) = \frac{1+sT}{sT}$

Soln:

(a) $G(s) = \frac{1}{1+sT}$ Type - 0 order 1

put $s = j\omega$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}\ \angle \tan^{-1}\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$



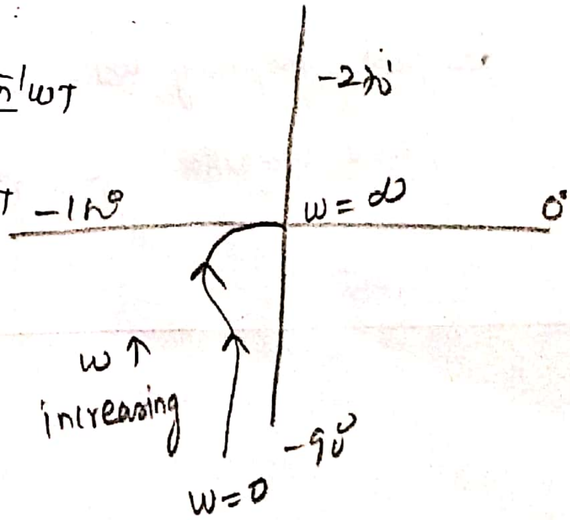
As $\omega \to 0$, $G(j\omega) \to 1 \angle 0°$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -180°$

(b) $G(s) = \frac{1}{s(1+sT)}$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)} = \frac{1}{\omega\angle 90°\sqrt{1+\omega^2T^2}\ \angle \tan^{-1}\omega T}$$

$$= \frac{1}{\omega\sqrt{1+\omega^2T^2}} \angle -90° - \tan^{-1}\omega T$$



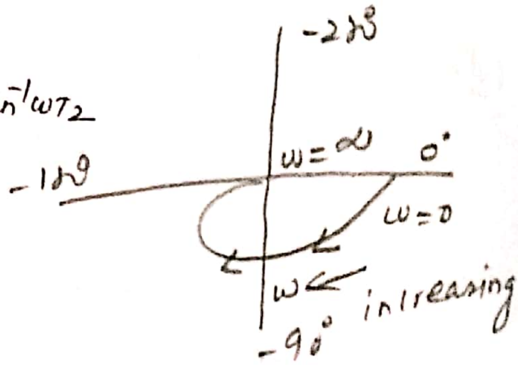
As $\omega \to 0$, $G(j\omega) \to \infty \angle -90°$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -180°$

(c) $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}$

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{1}{\sqrt{1+\omega^2T_1^2}\,\angle \tan^{-1}\omega T_1 \,\sqrt{1+\omega^2T_2^{\,2}}\,\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\sqrt{(1+\omega^2T_1^{\,2})(1+\omega^2T_2^{\,2})}}\,\angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

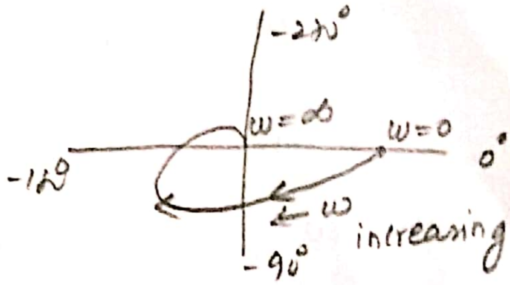


As $\omega \to 0$, $G(j\omega) \to 1\,\angle 0^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\,\angle -180^\circ$

(d) $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$



$$= \frac{1}{\sqrt{1+\omega^2T_1^2}\,\angle \tan^{-1}\omega T_1\,\sqrt{1+\omega^2T_2^2}\,\angle \tan^{-1}\omega T_2\,\sqrt{1+\omega^2T_3^2}\,\angle \tan^{-1}\omega T_3}$$

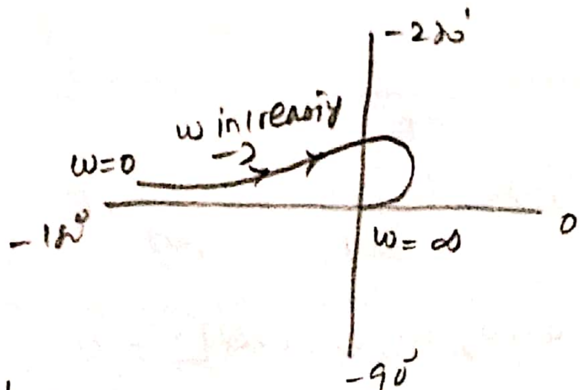
$$= \frac{1}{\sqrt{(1+\omega^2T_1^{\,2})(1+\omega^2T_2^{\,2})(1+\omega^2T_3^{\,2})}}\,\angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$$

As $\omega \to 0$, $G(j\omega) \to 1\,\angle 0^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\,\angle -270^\circ$

(e) $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$$



$$G(j\omega) = \frac{1}{\omega^2\,\angle +180^\circ\,\sqrt{1+\omega^2T_1^{\,2}}\,\angle \tan^{-1}\omega T_1\,\sqrt{1+\omega^2T_2^{\,2}}\,\angle \tan^{-1}\omega T_2}$$

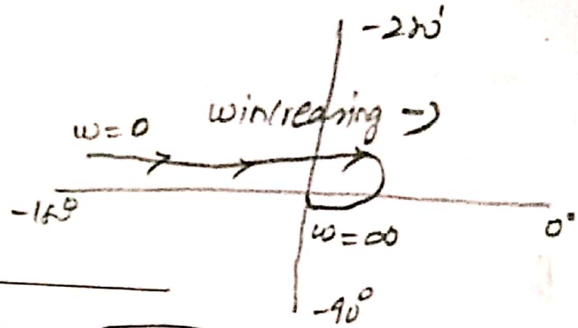
$$G(j\omega) = \frac{1}{\omega^2 \sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

As $\omega \to 0$, $G(j\omega) \to \infty \angle -180^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -360^\circ$

(f) $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$



$$= \frac{1}{\omega^2 \angle 180^\circ \sqrt{1+\omega^2 T_1^2}\ \tan^{-1}\omega T_1 \sqrt{1+\omega^2 T_2^2}\ \tan^{-1}\omega T_2 \sqrt{1+\omega^2 T_3^2}\ \tan^{-1}\omega T_3}$$

$$= \frac{1}{\omega^2 \sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$$

As $\omega \to 0$, $G(j\omega) \to \infty \angle -180^\circ$

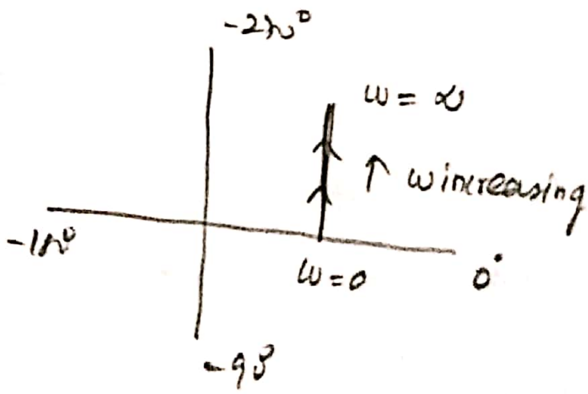
As $\omega \to \infty$, $G(j\omega) \to 0 \angle -450^\circ = 0 \angle -90^\circ$

(g) $G(s) = 1+sT$

$G(j\omega) = 1+j\omega T = 1+\omega T \angle 90^\circ$

As $\omega \to 0$, $G(j\omega) \to 1+0 \angle 90^\circ$

As $\omega \to \infty$, $G(j\omega) \to 1 \angle \infty \angle 90^\circ$

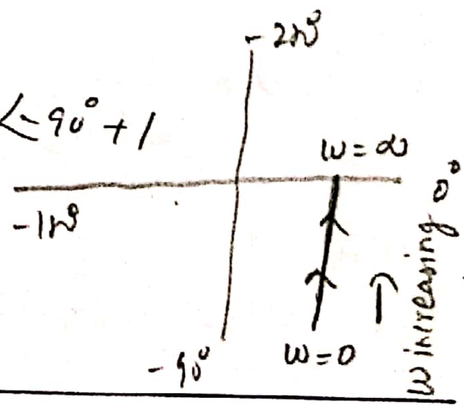


(h) $G(s) = \dfrac{1+sT}{sT}$

$$G(j\omega) = \frac{1+j\omega T}{j\omega T} = \frac{1}{j\omega T}+1 = \frac{1}{\omega T \angle 90^\circ}+1 = \frac{1}{\omega T}\angle -90^\circ + 1$$

As $\omega \to 0$, $G(j\omega) \to \infty \angle -90^\circ + 1$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -90^\circ + 1$



(7) i) Determine the closed loop bandwidth, closed loop peak magnitude, gain margin and phase margin for the following system

$$G(s) = \frac{4}{s(s+1)}.$$

Soln:

Given that $G(s) = \frac{4}{s(s+1)}$

C.L.T.F $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{4/s(s+1)}{1+\frac{4}{s(s+1)}\cdot 1}$

$$\frac{C(s)}{R(s)} = \frac{4}{s(s+1)+4} = \frac{4}{s^2+s+4} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$\omega_n^2 = 4$

$\omega_n = \sqrt{4} = 2$ rad/sec

$2\xi\omega_n = 1$

$\xi = \frac{1}{2\omega_n} = \frac{1}{2(2)} = 0.25$

closed loop bandwidth:

$$\omega_b = \omega_n\left[1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}\right]^{1/2}$$

$$= 2\left[1-2(0.25)^2+\sqrt{2-4(0.25)^2+4(0.25)^4}\right]^{1/2}$$

$\omega_b = 2.9689$ rad/sec

Closed loop peak magnitude

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2(0.25)\sqrt{1-(0.25)^2}} = 2.065$$

Gain margin

The gain margin of 2nd order system is infinite

$\therefore$ Gain margin $= \infty$

Phase margin

$$\gamma = 180^\circ + \tan^{-1}\left\{\frac{2\zeta}{\left[-2\zeta^2 + \sqrt{4\zeta^4+1}\,\right]^{1/2}}\right\}$$

$$= 180^\circ + \tan^{-1}\left(\frac{0.5}{0.9354}\right)$$

$$= 180^\circ + 28.12^\circ$$

$$\gamma = 208.12^\circ$$

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7. ii) Explain the use of Nichol's chart to obtain closed loop frequency response from open loop frequency response of a unity feedback system.

The chart obtained after transforming the constant M and N circles to log magnitude and phase angle co-ordinates is known as Nichol's chart. It is an ordinary graph sheet on which the M and N contours are superimposed. The M contours are the magnitude of closed loop system in dB and N contours are the corresponding phase angle locus of closed loop system. The magnitude M and phase angle ϕ will be constant along each M and N contour respectively. The ordinary graph sheet contains magnitude in dB along y axis and phase angle in degrees along x-axis.

The Nichol's plot of open loop system is a graph between magnitude of $G(j\omega)$ in dB and phase angle in degrees on an ordinary graph sheet. Nichol's plot is drawn by calculating the magnitude and phase of $G(j\omega)$ for various values of ω ie Bode plot of $G(j\omega)$ is sketched and for any frequency the magnitude and phase of $G(j\omega)$ can be obtained.

The closed loop frequency response can be determined graphically from the open loop frequency response using Nichol's chart.