01. Solve the following system of equations by Gauss Jacobi and Gauss Seidal method correct to three decimal places.

$x + y + 54z = 110, \quad 27x + 6y - z = 85, \quad 6x + 15y + 2z = 72$

Solution:-

The given equations can rewrite as;

$27x = 85 - 6y + z \quad ; \quad 6x + 15y + 2z = 72 \quad ; \quad x + y + 54z = 110$

$x = \dfrac{85 - 6y + z}{27} \qquad y = \dfrac{72 - 6x - 2z}{15} \qquad 54z = 110 - x - y$

$z = \dfrac{110 - x - y}{54}.$

(i) Gauss Jacobi method

Iteration	$x = \dfrac{85-6y+z}{27}$	$y = \dfrac{72-6x-2z}{15}$	$z = \dfrac{110-x-y}{54}$
01	$y=0; \; z=0$ $x = \dfrac{85}{27} = 3.1481$	$x=0; \; z=0$ $y = \dfrac{72}{15} = 4.8$	$x=0; \; y=0$ $z = \dfrac{110}{54} = 2.0370$
02.	$y=4.8, \; z=2.0370$ $x = 2.1569$	$x=3.1481, \; z=2.0370$ $y = 3.2691$	$x=3.1481, \; y=4.8$ $z = 1.8898$
03.	2.4916	3.6852	1.9365
04.	2.4009	3.5451	1.9226
05.	2.4315	3.5832	1.9269
06.	2.4232	3.5704	1.9256

The solution is, $x = 2.4232, \quad y = 3.5704, \quad z = 1.9256.$

(ii) Gauss Seidal method:.

$x = \dfrac{85 - 6y + z}{27}, \quad y = \dfrac{72 - 6x - 2z}{15}, \quad z = \dfrac{110 - x - y}{54}.$

Iteration	$x = \frac{85 - 6y + z}{27}$	$y = \frac{72 - 6x - 2z}{15}$	$z = \frac{110 - x - y}{54}$
01.	$y = 0; \quad z = 0$ $x = \frac{85}{27} = 3.1481$	$x = 3.1481, \quad z = 0$ $y = 3.5407$	$x = 3.1481, \quad y = 3.5407$ $z = 1.9131$
02.	$y = 3.5407, \quad z = 1.9131$ $x = 2.4321$	$x = 2.4321, \quad z = 1.9131$ $y = 3.5720$	$x = 2.4321, \quad y = 3.5720$ $z = 1.9258$
03.	2.4256	3.5729	1.9259
04.	2.4254	3.5730	1.9259
05.	2.4254	3.5730	1.9259

The solution is, $x = 2.4254$, $y = 3.5730$, $z = 1.9259$.

02. Find the dominant eigen value and the corresponding eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Solution:-

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an approximate eigen value.

$$AX_1 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot X_2$$

$$AX_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix}$$

$$AX_3 = \begin{pmatrix} 3.5714 \\ 1.8572 \\ 0 \end{pmatrix} = 3.5714\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix}$$

$$AX_4 = \begin{pmatrix} 4.12 \\ 2.04 \\ 0 \end{pmatrix} = 4.12 \begin{pmatrix} 1 \\ 0.4951 \\ 0 \end{pmatrix} = 4.12 X_5$$

$$AX_5 = \begin{pmatrix} 3.9706 \\ 1.9902 \\ 0 \end{pmatrix} = 3.9706 \begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = 3.9706\, X_6$$

$$AX_6 = \begin{pmatrix} 4.0072 \\ 2.0024 \\ 0 \end{pmatrix} = 4.0072 \begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = 4.0072\, X_7$$

$$AX_7 = \begin{pmatrix} 3.9982 \\ 1.9994 \\ 0 \end{pmatrix} = 3.9982 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 3.9982\, X_8$$

$$AX_8 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 4 X_9$$

Dominant eigen value $= 4$, Corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$

To find the least eigen value, Let $B = A - 4I$ since $\lambda_1 = 4$.

$$B = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

we will find the dominant eigen value of B

Let $Y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be the initial vector

$$BY_1 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = -3 Y_2$$

$$BY_2 = \begin{pmatrix} -5 \\ 1.6666 \\ 0 \end{pmatrix} = -5 \begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = -5 Y_3$$

$$BY_3 = \begin{pmatrix} -5 \\ 1.6666 \\ 0 \end{pmatrix} = -5 \begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix}$$

Dominant eigen value of $B = -5$.

Adding 4, smallest eigen value of $A = -5+4 = -1$.

Sum of eigen value $=$ Trace of the matrix $= 1+2+3 = 6$

$4 + -1 + \lambda_3 = 6$; $\lambda_3 = 3$. The eigen values are $4, 3, -1$.

03 Find the real Positive root of $3x - \cos x - 1 = 0$ by Newton's Raphson Method correct to three decimal Places.

Solution:

The given equation is, $f(x) = 3x - \cos x - 1 = 0$

$$f'(x) = 3 + \sin x$$

Now: $f(0) = 3(0) - \cos 0 - 1 = -2 = -ve$

$f(1) = 3(1) - \cos 1 - 1 = +ve$

The Root lies Between 0 and 1. Let $x_0 = 0.5$

The newton Raphson method is, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Put $n = 0$; $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{[3(0.5) - \cos(0.5) - 1]}{3 + \sin(0.5)}$$

$$= 0.6085$$

Put $n = 1$; $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 0.6085 - \frac{f(0.6085)}{f'(0.6085)}$$

$$= 0.6085 - \frac{[(3 \times 0.6085) - \cos(0.6085) - 1]}{3 + \sin(0.6085)}$$

$$= 0.6071$$

Put $n = 2$; $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.6071 - \frac{f(0.6071)}{f'(0.6071)}$$

$$= 0.6071.$$

The Root is 0.6071.

04. Find a Positive root of $3x - \sqrt{1+\sin x} = 0$ by Iteration method.

Solution:-

The given equation is $3x - \sqrt{1+\sin x} = 0$

$$3x = \sqrt{1+\sin x}$$

$$x = \frac{\sqrt{1+\sin x}}{3}$$

$$\therefore \varphi(x) = \frac{\sqrt{1+\sin x}}{3}$$

and $$\varphi'(x) = \frac{\cos x}{6\sqrt{1+\sin x}}$$

The root of given equation lies in $(0,1)$.

Since $f(0) = -ve$; and $f(1) = +ve$.

Let $x_0 = 0.4$: $x_1 = \frac{1}{3}\sqrt{1+\sin(0.4)} = 0.3929$

$$x_2 = 0.3919$$

$$x_3 = 0.3918$$

$$x_4 = 0.3918$$

The root is 0.3918.

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05 Solve the system by Gauss Elimination method

$2x+3y-z = 5$; $4x+4y-3z=3$; and $2x-3y+2z = 2$.

Solution:-

The system is equivalent $AX = B$ where $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$B = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$

The augmented matrix $(A,B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right)$

$$(A,B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right) \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \end{array}$$

$$(A,B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right) \quad R_3 = R_3 - 3R_2$$

By Back substitution method:

$6z = 18$; $z = \frac{18}{6} = 3$

and $-2y - z = -7$

$-2y - 3 = -7$; $-2y = -4$; $y = 2$

and $2x + 3y - z = 5$

$2x + 3(2) - 3 = 5$; $2x + 6 - 3 = 5$

$2x = 2$

$x = 1$.

The solution is, $x = 1$, $y = 2$, $z = 3$.

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06 Solve the system of equations $4x + y + 3z = 11$, $3x + 4y + 2z = 11$, $2x + 3y + z = 7$ by Gauss Jordan method.

Solution:

The system of equations can be rewrite as $AX = B$.

$$A = \begin{pmatrix} 4 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 11 \\ 11 \\ 7 \end{pmatrix}$$

The Augment matrix is, $(A,B) = \left(\begin{array}{ccc|c} 4 & 1 & 3 & 11 \\ 3 & 4 & 2 & 11 \\ 2 & 3 & 1 & 7 \end{array}\right)$

$$(A, B) = \left(\begin{array}{ccc|c} 4 & 1 & 3 & 11 \\ 0 & -13 & 1 & -11 \\ 0 & -5 & 1 & -3 \end{array}\right) \quad \begin{array}{l} R_2' = 3R_1 - 4R_2 \\ R_3' = R_1 - 2R_3 \end{array}$$

$$= \left(\begin{array}{ccc|c} 13 & 0 & 10 & 33 \\ 0 & -13 & 1 & -11 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1'' = \dfrac{13R_1' + R_2'}{4} \\ R_3'' = \dfrac{5R_2' - 13R_3'}{-8} \end{array}$$

$$= \left(\begin{array}{ccc|c} 13 & 0 & 0 & 13 \\ 0 & 13 & 0 & 13 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1''' = R_1'' - 10R_3'' \\ R_2''' = R_3'' - R_2'' \end{array}$$

$\therefore$ $13x = 13$, $x = 1$; $13y = 13$, $y = 1$; $z = 2$

The solution is, $x = 1$, $y = 1$, $z = 2$.

07. Find the Iterative for $\sqrt{N}$ by using Newton Raphson method and hence find the value of $\sqrt{5}$.

Solution:

Let $x = \sqrt{N}$

$x^2 = N$

$f(x) = x^2 - N = 0$

$f'(x) = 2x$

The newton Raphson method is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - N)}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}$$

To Find the value of $\sqrt{5}$:-

$f(x) = x^2 - 5 = 0$

$f(1) = 1 - 5 = -4$

$f(2) = 4 - 5 = -1$

$f(3) = 9 - 5 = 4$

Root lies Between 2 and 3

Let $x_0 = 2.5$

$$x_1 = \frac{x_0^2 + 5}{2x_0} = 2.25$$

$$x_2 = \frac{x_1^2 + 5}{2x_1} = 2.236$$

$$x_3 = \frac{x_2^2 + 5}{2x_2} = 2.236$$

The solution is $x = 2.236$.