ISO PARAMETRIC ELEMENTS

AND

NUMERICAL INTEGRATION ⇨

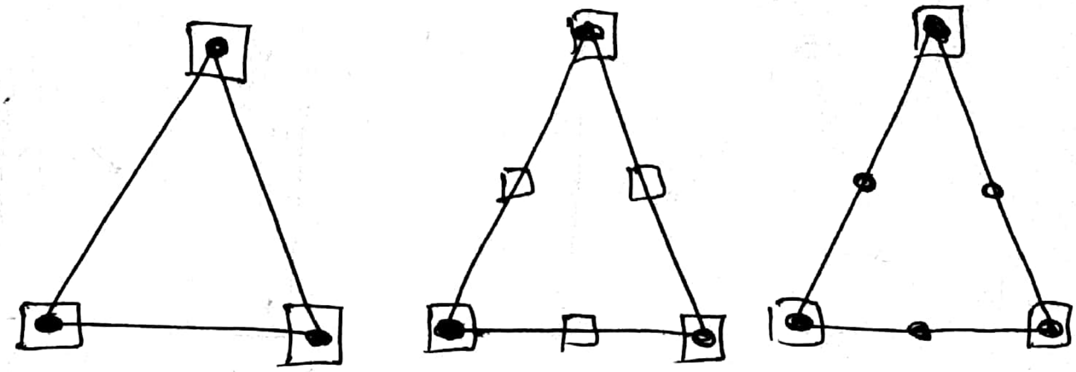
Mohan S R, AP/Mech

UNIT-5

→ Isoparametric (a)

→ Super parametric (b)

→ Sub parametric (c)



(a) (b) (c)

● → Nodes for defining displacements

□ → Nodes for defining geometry.

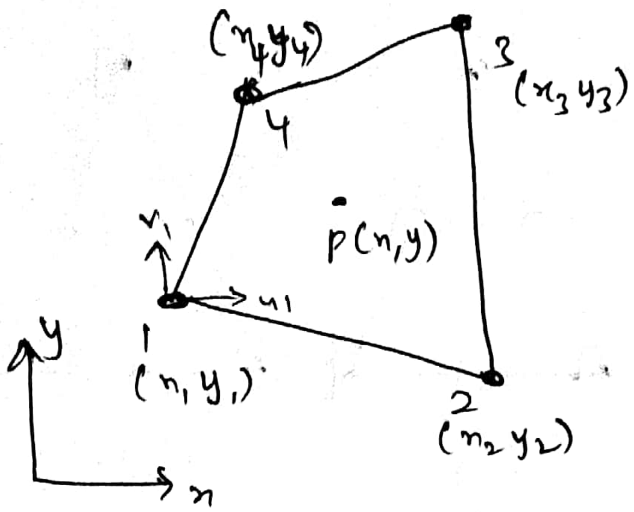
Higher order elements :

For any element, if the interpolation polynomial is of order two or more, the element is known as higher order elements.

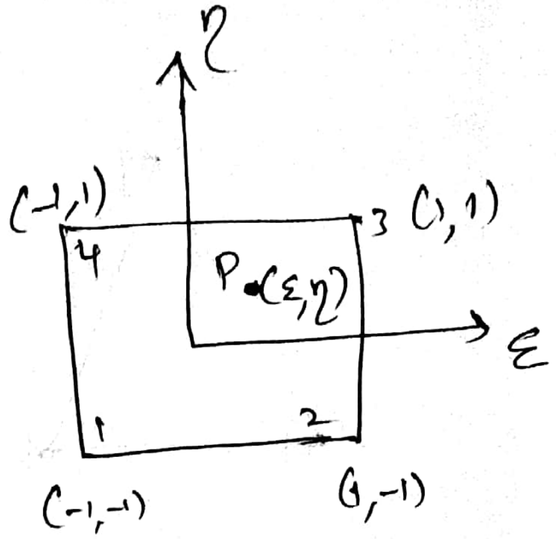
Exp:

$$\phi(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + \dots + a_m y^n.$$

Shape functions 4-noded Quadrilateral Elements (Natural Co-ordinates)



a) Quadrilateral element



b) Isoparametric element (Master element)

$$u(\varepsilon,\eta) = a_1 + a_2\varepsilon + a_3\eta + a_4\varepsilon\eta$$
$$v(\varepsilon,\eta) = a_5 + a_6\varepsilon + a_7\eta + a_8\varepsilon\eta.$$

by Solving,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta) \qquad N_2 = 1/4\,(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta) \qquad N_4 = 1/4\,(1-\varepsilon)(1+\eta)$$

&

$$N_1 + N_2 + N_3 + N_4 = 1$$

Jacobian Matrix

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4\right]$$

$$J_{22} = \frac{1}{4}\left[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4\right]$$

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① For the element shown in fig. determine the Jacobian Matrix.

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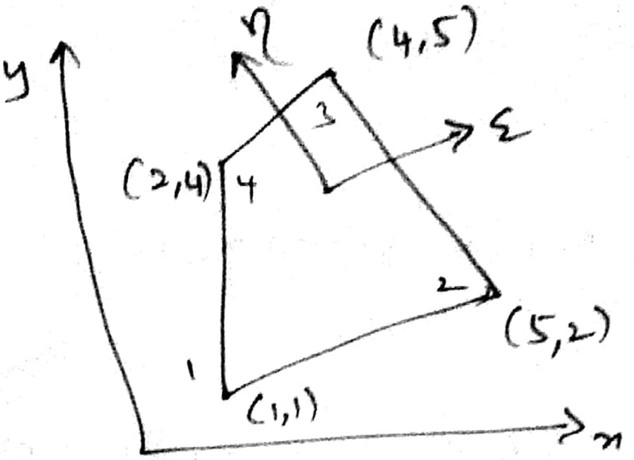

Sol:

$x_1 = 1 \qquad y_1 = 1$

$x_2 = 5 \qquad y_2 = 2$

$x_3 = 4 \qquad y_3 = 5$

$x_4 = 2 \qquad y_4 = 4.$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)\,x_1 + (1-\eta)\,x_2 + (1+\eta)\,x_3 - (1+\eta)\,x_4\right]$$

$$= \frac{1}{4}\left[-(1-\eta)\,1 + (1-\eta)\,5 + (1+\eta)\,4 - (1+\eta)\,2\right]$$

$$= \frac{1}{4}\left[-1+\eta+5-5\eta+4+4\eta-2-2\eta\right]$$

$$= \frac{1}{4}\left[-2\eta+6\right] = -0.5\eta+1.5$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$= \frac{1}{4}\left[-(1-\eta)1 + (1-\eta)2 + (1+\eta)5 - (1+\eta)4\right]$$

$$= \frac{1}{4}\left[-1+\eta+2-2\eta+5+5\eta-4-4\eta\right]$$

$$= \frac{1}{4}[2] = 0.5$$

$$J_{21} = \frac{1}{4}\left[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4\right]$$

$$= \frac{1}{4}\left[-(1-\varepsilon)1 - (1+\varepsilon)5 + (1+\varepsilon)4 + (1-\varepsilon)2\right]$$

$$= 1/4\left[-1+\varepsilon-5-5\varepsilon+4+4\varepsilon+2-2\varepsilon\right]$$

$$= 1/4[-2\varepsilon] = -0.5\varepsilon.$$

$$J_{22} = \frac{1}{4}\left[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4\right]$$

$$= 1/4\left[-(1-\varepsilon)1 - (1+\varepsilon)2 + (1+\varepsilon)5 + (1-\varepsilon)4\right]$$

$$= 1/4\left[-1+\varepsilon-2-2\varepsilon+5+5\varepsilon+4-4\varepsilon\right]$$

$$= 1/4[6] = 1.5$$

$$\therefore [J] = \begin{bmatrix} -0.5\eta+1.5 & 0.5 \\ -0.5\varepsilon & 1.5 \end{bmatrix}$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)\,y_1 + (1-\eta)\,y_2 + (1+\eta)\,y_3 - (1+\eta)\,y_4\right]$$

$$= \frac{1}{4}\left[-(1-\eta)\,1 + (1-\eta)\,0 + (1+\eta)\,2.5 - (1+\eta)\,3\right]$$

$$= \frac{1}{4}\left[-1+\eta + 2.5 + 2.5\eta - 3 - 3\eta\right]$$

$$= \frac{1}{4}\left[0.5\eta - 1.5\right]$$

$$= \frac{1}{4}\left[0.5\,(1/3) - 1.5\right]$$

$$J_{12} = -0.333$$

$$J_{21} = \frac{1}{4}\left[-(1-\varepsilon)x_1 - (1+\varepsilon)\,x_2 + (1+\varepsilon)\,x_3 + (1-\varepsilon)x_4\right]$$

$$= \frac{1}{4}\left[-(1-\varepsilon)\,1 - (1+\varepsilon)\,3 + (1+\varepsilon)\,3.5 + (1-\varepsilon)\,2\right]$$

$$= \frac{1}{4}\left(-1+\varepsilon - 3 - 3\varepsilon + 3.5 + 3.5\varepsilon + 2 - 2\varepsilon\right]$$

$$= \frac{1}{4}\left[-0.5\varepsilon + 1.5\right]$$

$$= \frac{1}{4}\left[-0.5(1/3) + 1.5\right] = 0.333$$

$$J_{22} = \frac{1}{4}\left[-(1+\varepsilon)\,y_1 - (1+\varepsilon)\,y_2 + (1+\varepsilon)\,y_3 + (1-\varepsilon)\,y_4\right]$$

$$= \frac{1}{4}\left[-(1-\varepsilon) - (1+\varepsilon)\,0 + (1+\varepsilon)\,2.5 + (1-\varepsilon)\,3\right]$$

$$= \frac{1}{4}\left[-1+\varepsilon + 2.5 + 2.5\varepsilon + 3 - 3\varepsilon\right] = \frac{1}{4}\left[0.5\varepsilon + 4.5\right]$$

$$J_{22} = \frac{1}{4}\left[0.5(1/3) + 4.5\right) = 1.16$$

$$\therefore [J] = \begin{bmatrix} 0.83 & -0.33 \\ 0.33 & 1.16 \end{bmatrix}$$

③ Evaluate the integral by two point Gaussian Quadrature $I = \int_{-1}^{1}\int_{-1}^{1} (2x^2 + 3xy + 4y^2)\, dx . dy$. Gauss points are $+0.57735$ and -0.57735 each of weight 1.000.

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Sol:

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\, dx\, dy = \sum_{j=1}^{n}\sum_{i=1}^{n} w_j . w_i f(x_i, y_i) \ldots$$

$$f(x,y) = 2x^2 + 3xy + 4y^2.$$

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\, dx\, dy = w_1^2 (f x_1, y_1) + w_1 w_2 f(x_1, y_2)$$
$$+ w_2^2 (f x_2, y_2) + w_1 w_2 f(x_2, y_1)$$

for 2 points,

$$w_1 = w_2 = 1.0$$

$$x_1, y_1 = +0.57735, +0.57735$$

$$x_2, y_2 = -0.57735, -0.57735$$

$$w_1^2 f(x_1, y_1) = w_1^2 (2x_1^2 + 3x_1 y_1 + 4y_1^2)$$

$$= (\cancel{0.57735}\ 1.0)^2 (2(0.57735)^2 + 3(0.57735)(0.57735) + 4(0.57735)^2$$

$$= 3.0$$

$$w_1 w_2 f(x_1, y_2) = w_1 w_2 (2x_1^2 + 3x_1 y_2 + 4y_2^2)$$

$$= 1.0(1.0)(2(0.57735)^2 + 3(0.57735)(-0.57735) + 4(-0.57735)^2)$$

$$= 1.0$$

$$w_2 w_1 f(x_2, y_1) = w_2 w_1 (2x_2^2 + 3x_2 y_1 + 4y_1^2)$$

$$= 1.0(1.0)\left[2(-0.57735)^2 + 3(-0.57735)(0.57735) + 4(0.57735)^2\right]$$

$$= 1.0$$

$$w_2^2 f(x_2, y_2) = w_2^2 (2x_2^2 + 3x_2 y_2 + 4y_2^2)$$

$$= 1.0^2\left[2(-0.57735)^2 + 3(-0.57735)(-0.57735) + 4(-0.57735)^2\right]$$

$$= 3.0$$

Hence,

$$\int_{-1}^{1}\int_{-1}^{1} (2x^2 + 3xy + 4y^2)\,dx\,dy = 3 + 1 + 1 + 3$$

$$= 8.0 //.$$

(4) Using Gauss Quadrature evaluate the following integral.

$$I = \int_{-1}^{1} (4\varepsilon^3 - 2\varepsilon^2 + 3\varepsilon + 6)\, d\varepsilon.$$

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Sol:

To find no of gauss points, 3-degree of fn.

let $2n-1 = 3$.

$n = \frac{3+1}{2} = 2$ points.

$\therefore \quad W_1 = W_2 = 1.0$

$\varepsilon_1 = \varepsilon_2 = \pm 0.57735.$

$$\int_{-1}^{1} f(x)\,dx = W_1 f(x_1) + W_2 f(x_2)$$

$$W_1 f(\varepsilon_1) = W_1 (4\varepsilon_1^3 - 2\varepsilon_1^2 + 3\varepsilon_1 + 6)$$

$$= 1 \left[4(0.57735)^3 - 2(0.57735)^2 + 3(0.57735) + 6 \right)$$

$$= 7.83$$

$$W_2 f(\varepsilon_2) = W_2 (4\varepsilon_2^3 - 2\varepsilon_2^2 + 3\varepsilon_2 + 6)$$

$$= 1\left[4(-0.57735)^3 - 2(-0.57735)^2 + 3(-0.57735) + 6 \right]$$

$$= 4.16$$

$$I = 7.83 + 4.16 = 11.99 \approx 12.$$

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(5) Using Gauss Quadrature evaluate the following integral.

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{12+2\varepsilon^2+\eta}{1+\eta^2}\, d\varepsilon\, d\eta .$$

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Sol:

$$f(x,y) = \frac{12+2\varepsilon^2+\eta}{1+\eta^2}\, d\varepsilon d\eta .$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\varepsilon,\eta)\, d\varepsilon d\eta = w_1^2 f(\varepsilon_1,\eta_1) + w_2^2 f(\varepsilon_2,\eta_2) + w_1 w_2 f(\varepsilon_1,\eta_2) + w_1 w_2 f(\varepsilon_2,\eta_1)$$

For 2 point scheme,

$$w_1 = w_2 = 1.0$$

$$\varepsilon_1, \eta_1 = 0.57735,\ 0.57735$$

$$\varepsilon_2, \eta_2 = -0.57735,\ -0.57735$$

$$w_1^2 f(\varepsilon_1,\eta_1) = w_1^2\left(\frac{12+2\varepsilon_1^2+\eta_1}{1+\eta_1^2}\right)$$

$$= 1^2\left(\frac{12+2(0.57735)^2+0.57735}{1+0.57735^2}\right) = 9.93$$

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$$W_2^2 f(\varepsilon_2, \eta_2) = W_2^2 \left[\frac{12 + 2\varepsilon_2^2 + \eta_2}{1+\eta_2^2} \right]$$

$$= 1^2 \left[\frac{12 + 2(-0.57735)^2 + (-0.57735)}{1 + (-0.57735)^2} \right]$$

$$= \cancel{81.86} = 9.09$$

$$W_1 W_2 f(\varepsilon_1, \eta_2) = W_1 W_2 \left(\frac{12 + 2\varepsilon_1^2 + \eta_2}{1+\eta_2^2} \right)$$

$$= 1 \times 1 \left[\frac{12 + 2(0.57735)^2 + (-0.57735)}{1 + (-0.57735)^2} \right]$$

$$= \cancel{81.63} = 9.09$$

$$W_1 W_2 f(\varepsilon_2, \eta_1) = W_1 W_2 \left(\frac{12 + 2\varepsilon_2^2 + \eta_1}{1+\eta_1^2} \right)$$

$$= 1 \times 1 \left[\frac{12 + 2(-0.57735)^2 + 0.57735}{1 + (0.57735)^2} \right]$$

$$= 9.93$$

$$\therefore I = 9.93 + 9.09 + 9.09 + 9.93$$

$$I = 38.04 \,//$$