TWO DIMENTIONAL

AXI - SYMMETRIC COMPONENTS ⇨

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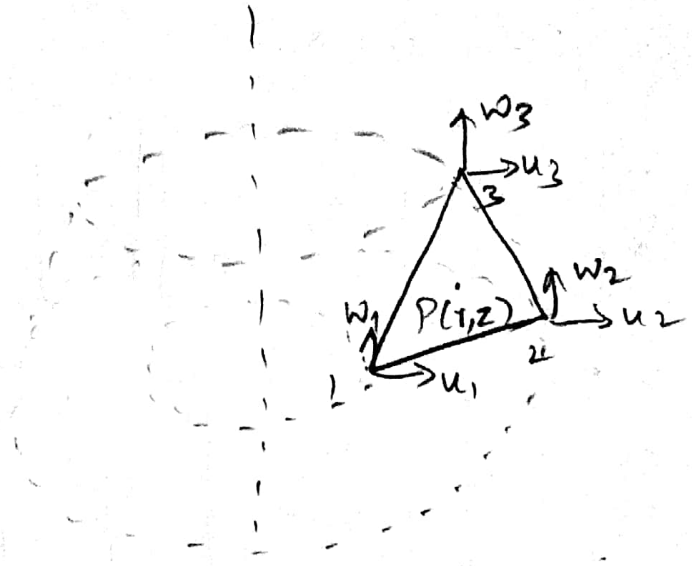
Derivation of Shape functions for Axisymmetric Triangular Element



Consider Axisymmetric triangular element.

Let

$u_1, w_1, u_2, w_2, u_3, w_3$ be nodal displacements

$u_1, u_2, u_3 \rightarrow$ radial direction (r)

$w_1, w_2, w_3 \rightarrow$ Axial direction (z)

$$\{\delta\} = \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{bmatrix}$$

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From polynomial eqn,

$$u(r,z) = a_1 + a_2 r + a_3 z$$

$$v(r,z) = a_4 + a_5 r + a_6 z$$

in Matrix, under $u \rightarrow$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$[a] = [D]^{-1}[u] \quad \text{--- (1)}$$

where $D \rightarrow$ Coordinate Matrix.

$$D^{-1} = \frac{[C]^T}{|D|}.$$

$C \rightarrow$ Co-factor Matrix.

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{ij} = (-1)^{i+j} |D_m|$$

$\hookrightarrow$ Minor matrix of $|D|$.

$$c_{11} = (-1)^{1+1} \begin{vmatrix} r_2 & z_2 \\ r_3 & z_3 \end{vmatrix} = r_2 z_3 - r_3 z_2 .$$

Similarly calculate for remaining factors,

$$\therefore [C] = \begin{bmatrix} r_2 z_3 - r_3 z_2 & z_2 - z_3 & r_3 - r_2 \\ r_3 z_1 - r_1 z_3 & z_3 - z_1 & r_1 - r_3 \\ r_1 z_2 - r_2 z_1 & z_1 - z_2 & r_2 - r_1 \end{bmatrix}$$

$$[C]^T = \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix}$$

$$|D| = \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$|D| = 2A$$

From eqn (1),

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$\alpha_1 = r_2 z_3 - r_3 z_2 \quad \alpha_2 = r_3 z_1 - r_1 z_3 \quad \alpha_3 = r_1 z_2 - r_2 z_1$

$\beta_1 = z_2 - z_3 \quad \beta_2 = z_3 - z_1 \quad \beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2 \quad \gamma_2 = r_1 - r_3 \quad \gamma_3 = r_2 - r_1$

WKT,

$$u(r,z) = a_1 + a_2 r + a_3 z.$$

$$u(r,z) = [1 \ r \ z] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$u(r,z) = [1 r z] \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A} & \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A} & \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 .$$

Similarly for, $w(r,z)$.

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3 .$$

Finally displacement,

$$\delta_p = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{bmatrix}$$

$$[\delta]_p = [N] . [\delta]$$

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Formula

* Stress-Strain relationship Matrix,

$$[\sigma] = [D].[e]$$

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{bmatrix} = [D] \begin{bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{bmatrix}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$\sigma_r \rightarrow$ Radial Stress

$\sigma_\theta \rightarrow$ Circumferential Stress

$\sigma_z \rightarrow$ Axial Stress

$\tau_{rz} \rightarrow$ Shear Stress.

Similarly for Strain $[e]$.

* Strain - Displacement Matrix

$$[e] = [B][\delta]$$

$$[\delta] = \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{bmatrix}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} & 0 & \frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} & 0 & \frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$r = \frac{r_1+r_2+r_3}{3}$$

$$z = \frac{z_1+z_2+z_3}{3}.$$

* Stress - Displacement Matrix.

$$[\sigma] = [D][e]$$

$$[\sigma] = [D][B][\delta]$$

* Stiffness Matrix:

$$[K] = [B]^T [D][B]\, 2\pi r . A$$

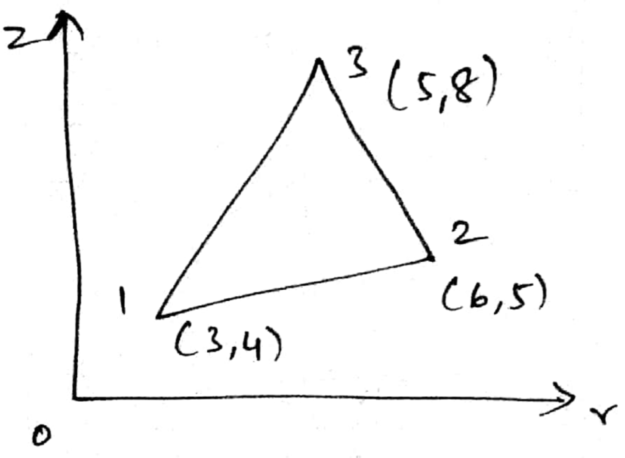
① Compute the Strain-displacement Matrix for the axisymmetric triangular element shown in fig. Also determine the element Strains. The nodal displacements are found out as

$u_1 = 0.002$ $\qquad$ $w_1 = 0.001$

$u_2 = 0.001$ $\qquad$ $w_2 = -0.004$

$u_3 = -0.003$ $\qquad$ $w_3 = 0.007$

All dimensions are in cm.



Sol:

$r_1 = 3$ $\qquad$ $z_1 = 4$

$r_2 = 6$ $\qquad$ $z_2 = 5$

$r_3 = 5$ $\qquad$ $z_3 = 8$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} & 0 & \frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} & 0 & \frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 5 \\ 1 & 5 & 8 \end{vmatrix}$$

$$A = 5\,cm^2$$

$\alpha_1 = r_2 z_3 - r_3 z_2 = (6\times8) - (5\times5) = 23$

$\alpha_2 = r_3 z_1 - r_1 z_3 = 5(4) - 3(8) = -4$

$\alpha_3 = r_1 z_2 - r_2 z_1 = 3(5) - 6(4) = -9$

$\beta_1 = z_2 - z_3 = 5-8 = -3$

$\beta_2 = z_3 - z_1 = 8-4 = 4$

$\beta_3 = z_1 - z_2 = 4-5 = -1$

$\gamma_1 = r_3 - r_2 = 5-6 = -1$

$\gamma_2 = r_1 - r_3 = 3-5 = -2$

$\gamma_3 = r_2 - r_1 = 6-3 = 3.$

Coordinate $\quad r = \dfrac{r_1+r_2+r_3}{3} = \dfrac{3+6+5}{3} = 4.7$

$$z = \frac{z_1+z_2+z_3}{3} = \frac{4+5+8}{3} = 5.7$$

Now,

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = 0.68$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = 0.72$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = 0.72.$$

Hence,

$$B = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.68 & 0 & 0.72 & 0 & 0.72 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}$$

$$[e] = [B]\{\delta\}$$

$$\begin{bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.68 & 0 & 0.72 & 0 & 0.72 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}\begin{bmatrix} 0.002 \\ 0.001 \\ 0.001 \\ -0.004 \\ -0.003 \\ 0.007 \end{bmatrix}$$

$$\begin{bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} 0.0001 \\ -0.000008 \\ 0.0028 \\ -0.0039 \end{bmatrix}$$

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② Calculate the element Stresses for the axisymmetric element shown in fig. The nodal displacements are,

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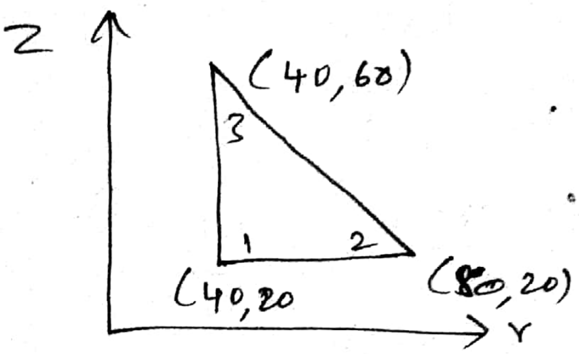
$u_1 = 0.02$ mm $w_1 = 0.03$ mm

$u_2 = 0.01$ mm $w_2 = 0.06$ mm

$u_3 = 0.04$ mm $w_3 = 0.01$ mm

Take $E = 210$ GPa $\mu = 0.25$



Sol:

$r_1 = 40$ $z_1 = 20$

$r_2 = 80$ $z_2 = 20$

$r_3 = 40$ $z_3 = 60$

WKT,

$$\{\sigma\} = [D][B][\delta]$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= \frac{210\times10^3}{(1+0.25)(1-0.5)} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[D] = 84\times10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} & 0 & \frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} & 0 & \frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\cancel{\alpha_1 r + \beta_1 r}$

$\alpha_1 = r_2 z_3 - r_3 z_2 = 80(60) - 40(20) = 4000$

$\alpha_2 = r_3 z_1 - r_1 z_3 = 40(20) - 40(60) = -1600$

$\alpha_3 = r_1 z_2 - r_2 z_1 = 40(20) - 80(20) = -800$

$\beta_1 = z_2 - z_3 = 20 - 60 = -40$

$\beta_2 = z_3 - z_1 = 60 - 20 = 40$

$\beta_3 = z_1 - z_2 = 20 - 20 = 0$

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$$\gamma_1 = r_3 - r_2 = 40 - 80 = -40$$

$$\gamma_2 = r_1 - r_3 = 40 - 40 = 0$$

$$\gamma_3 = r_2 - r_1 = 80 - 40 = 40.$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & 40 & 20 \\ 1 & 80 & 20 \\ 1 & 40 & 60 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(4800 - 800) - 40(60 - 20) + 20(40 - 80)\right]$$

$$A = 800 \text{ mm}^2$$

$$B = \frac{1}{2 \times 800}\begin{bmatrix} -40 & 0 & 40 & 0 & 0 & 0 \\ \frac{4800 - 40(53.3) + -40(33.3)}{53.3} & 0 & \frac{-1600 + 40(53.3) + 0}{53.3} & 0 & \frac{-800 + 0 + 40(33.3)}{53.3} & 0 \\ 0 & -40 & 0 & 0 & 0 & 40 \\ -40 & -40 & 0 & 40 & 40 & 0 \end{bmatrix}$$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{40 + 80 + 40}{3} = 53.33$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{20 + 20 + 60}{3} = 33.33$$

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$$B = \frac{1}{1600}\begin{bmatrix} -40 & 0 & 40 & 0 & 0 & 0 \\ 10 & 0 & 10 & 0 & 10 & 0 \\ 0 & -40 & 0 & 0 & 0 & 40 \\ -40 & -40 & 0 & 40 & 40 & 0 \end{bmatrix}$$

$$B = \frac{1}{160}\begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -4 & 0 & 4 & 4 & 0 \end{bmatrix}$$

$$\sigma = D B \delta.$$

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{bmatrix} = \frac{84\times10^3}{160}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -4 & 0 & 4 & 4 & 0 \end{bmatrix}\begin{bmatrix} 0.02 \\ 0.03 \\ 0.01 \\ 0.06 \\ 0.04 \\ 0.01 \end{bmatrix}$$

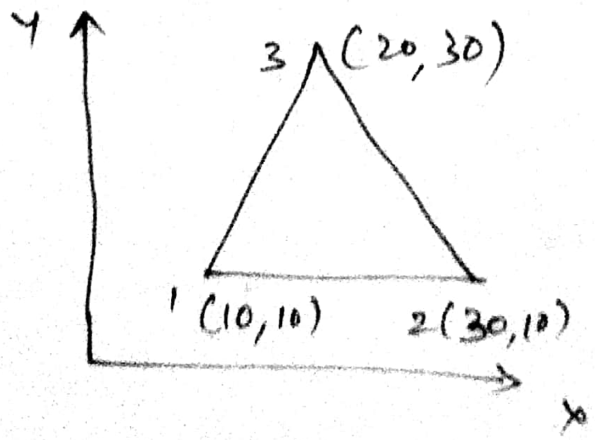
$$= \frac{84\times10^3}{160}\begin{bmatrix} -11 & -4 & 13 & 0 & 1 & 4 \\ -1 & -4 & 7 & 0 & 3 & 4 \\ -3 & -12 & 5 & 0 & 1 & 12 \\ -4 & -4 & 0 & 4 & 4 & 0 \end{bmatrix}\begin{bmatrix} 0.02 \\ 0.03 \\ 0.01 \\ 0.06 \\ 0.04 \\ 0.01 \end{bmatrix}$$

$$= 525 \begin{bmatrix} -0.13 \\ 0.08 \\ -0.21 \\ 0.2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} -68.25 \\ 42 \\ -110.25 \\ 105 \end{bmatrix} N/mm^2.$$

③ With at least two examples explain what is meant by axisymmetric analysis. For the 3 noded triangular axisymmetric element shown in fig. derive the strain displacement Matrix [B] and also Constitutive Matrix [D].

Apr/May 2018



Sol:

In Some 3D solids like cylinder, flywheel, turbine, discs etc, the material content is symmetric with respect to their axes. (i.e, material content is equal in equal distances of opposite sides) Hence the stress developed, displacement produced etc. are considered as symmetric. Such elements are known Axisymmetric elements.

Due to Axisymmetric Condition, these three dimensional solids can be treated as two dimensional solids for analysis.

$r_1 = 10$ $\quad z_1 = 10$

$r_2 = 30$ $\quad z_2 = 10$

$r_3 = 20$ $\quad z_3 = 30$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{10 + 30 + 20}{3} = 20$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{10 + 10 + 30}{3} = 16.67$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= \frac{210 \times 10^3}{(1+0.25)(1-2\times 0.5)} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[D] = 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} & 0 & \frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} & 0 & \frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = r_2 z_3 - r_3 z_2 = 30(30) - 20(10) = 700$

$\alpha_2 = r_3 z_1 - r_1 z_3 = 20(10) - 10(30) = -100$

$\alpha_3 = r_1 z_2 - r_2 z_1 = 10(10) - 30(10) = -200$

$$\beta_1 = z_2 - z_3 = 10 - 30 = -20$$

$$\beta_2 = z_3 - z_1 = 30 - 10 = 20$$

$$\beta_3 = z_1 - z_2 = 10 - 10 = 0.$$

$$\gamma_1 = r_3 - r_2 = 20 - 30 = -10$$

$$\gamma_2 = r_1 - r_3 = 10 - 20 = -10$$

$$\gamma_3 = r_2 - r_1 = 30 - 10 = 20.$$

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = \frac{700 + (-20)(20) + (-10)\,16.67}{20} = 6.66$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = \frac{-100 + 20(20) - 10(16.67)}{20} = 6.66.$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = \frac{-200 + 0 + 20(16.67)}{20} = 6.66$$

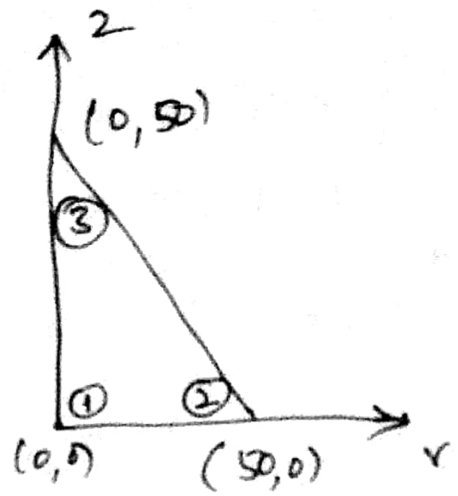
$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 10 & 10 \\ 1 & 30 & 10 \\ 1 & 20 & 30 \end{vmatrix} = \frac{1}{2}\left[1(700) - 10(20) + 10(-10)\right]$$

$$A = 200\ mm^2.$$

$$[B] = \frac{1}{400}\begin{bmatrix} \cancel{700} & \cancel{0} & \cancel{-100} & \cancel{0} & \cancel{-200} & \cancel{0} \\ -20 & 0 & 20 & 0 & 0 & 0 \\ 6.66 & 0 & 6.66 & 0 & 6.66 & 0 \\ 0 & -10 & 0 & -10 & 0 & 20 \\ -10 & -20 & -100 & 20 & 20 & 0 \end{bmatrix}$$

(4) For the axisymmetric element shown in fig. determine the Stiffness matrix. Let $E = 2.1 \times 10^5\ N/mm^2$ & $\mu = 0.25$. The Co-ordinates are in mm.

Nov/Dec 2016.



Sol:

$r_1 = 0 \qquad z_1 = 0$

$r_2 = 50 \qquad z_2 = 0$

$r_3 = 0 \qquad z_3 = 50$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{50}{3} = 16.66$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{50}{3} = 16.66$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 50 & 0 \\ 1 & 0 & 50 \end{vmatrix}$$

$$= \frac{1}{2}(2500) = 1250\ mm^2$$

$$D = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= \frac{2.1\times10^5}{(1+0.25)(1-2(0.25))} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$(D) = 84\times10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1+\beta_1 r+\gamma_1 z}{r} & 0 & \frac{\alpha_2+\beta_2 r+\gamma_2 z}{r} & 0 & \frac{\alpha_3+\beta_3 r+\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = 0$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = 0$$

$$\alpha_3 = r_1 z_1 - r_2 z_1 = 0$$

$$\beta_1 = z_2 - z_3 = -50$$

$$\beta_2 = z_3 - z_1 = 50$$

$$\beta_3 = z_1 - z_2 = 0$$

$$\gamma_1 = \gamma_3 - \gamma_2 = -50$$

$$\gamma_2 = \gamma_1 - \gamma_3 = 0$$

$$\gamma_3 = \gamma_2 - \gamma_1 = 50$$

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = \frac{0 + (-50)\,16.66 + (-50)\,16.66}{16.66} = -100$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = \frac{0 + 50(16.66) + 0}{16.66} = 50$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = \frac{0 + 0 + 50\,(16.66)}{16.66} = 50$$

$$[B] = \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ -100 & 0 & 50 & 0 & 50 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & -50 & 0 & 50 & 50 & 0 \end{bmatrix}$$

$$[B] = 50 \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$[B]^T = 50 \begin{bmatrix} -1 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Stiffness Matrix,

$$[K] = [B]^T [D] [B] \cdot 2\pi r \cdot A$$

$$[D][B] = 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} 50 \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\qquad\qquad 4\times4 \qquad\qquad 4\times6$$

$$= 42 \times 10^5 \begin{bmatrix} -5 & -1 & 4 & 0 & 1 & 1 \\ -7 & -1 & 4 & 0 & 3 & 1 \\ -3 & -3 & 2 & 0 & 1 & 3 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$[K] = 50 \begin{bmatrix} -1 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} 42\times10^{5} \begin{bmatrix} -5 & -1 & 4 & 0 & 1 & 1 \\ -7 & -1 & 4 & 0 & 3 & 1 \\ -7 & -3 & 2 & 0 & 1 & 3 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} \times 2\pi \times 13.16 \times 12500$$

$$(6\times6) \qquad (4\times6)$$

$$[K] = 2.7\times10^{13} \begin{bmatrix} 20 & 4 & -12 & -1 & -8 & -3 \\ 4 & 4 & -2 & -1 & -2 & -3 \\ -12 & -2 & 8 & 0 & 4 & 2 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -8 & -2 & 4 & 1 & 4 & 1 \\ -3 & -3 & 2 & 0 & 1 & 3 \end{bmatrix}$$

The above found Stiffness matrix is Correct.

Because of

* Any Column Sum is zero
* Sum of any row is also zero.
* Symmetrical Matrix.