TWO DIMENSIONAL

SCALAR / VECTOR VARIABLE $\Longrightarrow$

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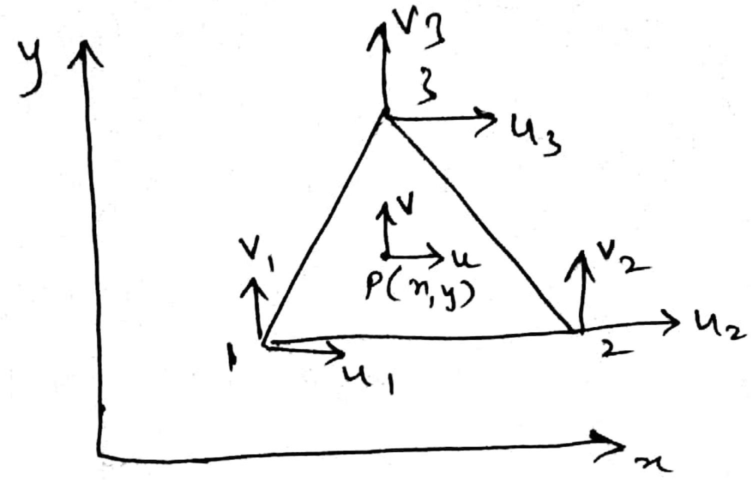
Derivation of Shape functions for Two dimensional Linear Element :



Consider CST element,

$\delta \rightarrow$ displacement

$u \rightarrow$ displacement Components in x axis

$v \rightarrow$ " in y axis

$x, y \rightarrow$ Co-ordinates of nodes.

polynomial eqn,

$$u(x,y) = a_1 + a_2 x + a_3 y$$

$$v(x,y) = a_4 + a_5 x + a_6 y .$$

apply nodal Conditions,

$$u = u_1 , \; v = v_1 , \; x = x_1 , \; y = y_1$$

$$u = u_2 , \; v = v_2 , \; x = x_2 , \; y = y_2$$

$$u = u_3 , \; v = v_3 , \; x = x_3 , \; y = y_3$$

So,

$$u_1 = a_1 + a_2 x_1 + a_3 y_1$$

$$u_2 = a_1 + a_2 x_2 + a_3 y_2$$

$$u_3 = a_1 + a_2 x_3 + a_3 y_3$$

&

$$v_1 = a_4 + a_5 x_1 + a_6 y_1$$

$$v_2 = a_4 + a_5 x_2 + a_6 y_2$$

$$v_3 = a_4 + a_5 x_3 + a_6 y_3 .$$

in matrix,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

(or)

$$[a] = [D^{-1}] [u]$$

$[D] \rightarrow$ Co-ordinate matrix.

$$D^{-1} = \frac{C^T}{|D|}$$

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

↳ Co-factor matrix.

$$C_{ij} = (-1)^{i+j} \cdot |D_m|$$

↳ Minor matrix,

$$C_{11} = (-1)^{1+1} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = x_2 y_3 - x_3 y_2 .$$

Similarly calculate for all co-factors.

$$[C] = \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

$$[C]^T = \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - 1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

$$|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$|D| = 2A \text{ [Triangular area) }.$$

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$$\therefore \quad [a] = [D^{-1}]\{u\}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

where,

$\alpha_1 = x_2y_3 - x_3y_2$

$\alpha_2 = x_3y_1 - x_1y_3$

$\alpha_3 = x_1y_2 - x_2y_1$

$\beta_1 = y_2 - y_3$

$\beta_2 = y_3 - y_1$

$\beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2$

$\gamma_2 = x_1 - x_3$

$\gamma_3 = x_2 - x_1$

$$\therefore \; u(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$u(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 + \beta_1 x + \gamma_1 y & \alpha_2 + \beta_2 x + \gamma_2 y & \alpha_3 + \beta_3 x + \gamma_3 y \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u(x,y) = N_1 u_1 + N_2 u_2 + N_3 u_3 .$$

where,

$$N_1 = \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A}$$

$$N_2 = \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A} .$$

Similarly for, $v(x, y)$,

$$v(x,y) = N_1 v_1 + N_2 v_2 + N_3 v_3 .$$

finally,

displacement,

$$\delta(x,y) = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$\delta(x,y) = [N][\delta]$$

<u>Note:</u>

$$N_1 + N_2 + N_3 = 1 .$$

Formulae :

* **Stress - Strain relation,**

$$[\sigma] = [D][e].$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [D] \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix}$$

$$D = \left[\frac{E}{1-\mu^2}\right] \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \text{ (plane Stress)}$$

$$D = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \text{ (plane Strain)}$$

$\mu \rightarrow$ poison ratio

$E \rightarrow$ young's modulus

$\sigma_x \rightarrow$ Stress in x axis

$\sigma_y \rightarrow$ " in y axis

$\tau_{xy} \rightarrow$ Shear Stress.

$e \rightarrow$ Strain.

* Strain - displacement relation

$$[e] = [B][\delta]$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_2 \end{bmatrix}$$

$$[\delta] = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}, \quad e = \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix},$$

* Stress - Displacement relation

$$[\sigma] = [D][e]$$

$$[\sigma] = [D][B][\delta]$$

Maximum normal stress,

$$\sigma_1 = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

Minimum normal stress,

$$\sigma_2 = \frac{1}{2}\left[(\sigma_x + \sigma_y) - \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

Maximum shear Stress,

$$\tau_{max} = \frac{1}{2} (\sigma_1 - \sigma_2)$$

principle angle θ_p,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}.$$

* **Stiffness Matrix** [K]

$$[K] = [B]^T [D] [B] \, A \cdot t.$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$t \rightarrow$ Thickness of element.

* **Temperature effects**:

$$e_0 = \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix} \text{ (plane Stress)}$$

$$e_0 = (1+\mu) \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix} \text{ (plane Strain)}$$

$\alpha \rightarrow$ Thermal expansion co-efficient

$\Delta T \rightarrow$ change in temperature.

$$[F_0] = [B]^T [D] [e_0] A \cdot t.$$

$$(6\times1) = (6\times3)(3\times3)(3\times1)$$

$$(6\times1) = (6\times1)$$

$$F_0 = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

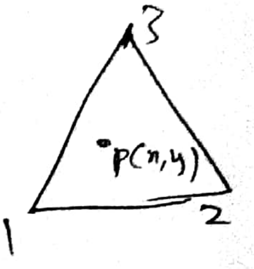
Note:

$$[F] = B^T D e_0 At$$

$$= B^T D B \delta At$$

$$[F] = [K] . [\delta]$$

Isoparametric Representation:



WKT,

$$u = N_1u_1 + N_2u_2 + N_3u_3$$

$$v = N_1v_1 + N_2v_2 + N_3v_3.$$

Co-ordinate of point P,

$$x = N_1x_1 + N_2x_2 + N_3x_3$$

$$y = N_1y_1 + N_2y_2 + N_3y_3.$$

$$(N_1 + N_2 + N_3 = 1), \quad N_3 = 1 - N_1 - N_2.$$

Substitute,

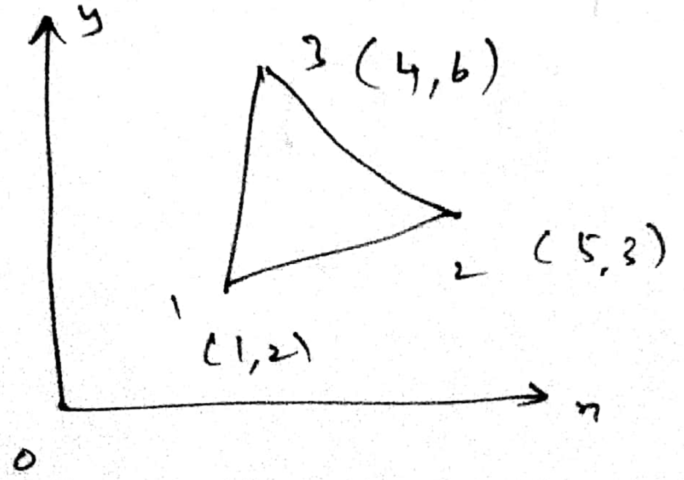
$$x = N_1 x_1 + N_2 x_2 + (1 - N_1 - N_2) x_3$$

$$\boxed{x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + x_3}$$

Similarly,

$$\boxed{y = (y_1 - y_3) N_1 + (y_2 - y_3) N_2 + y_3}$$

① The nodal co-ordinates of a triangular element are shown in fig. At the point P inside the element, the x co-ordinate is 3.3 and the shape function $N_1 = 0.3$. Determine the shape functions N_2, N_3 & y co-ordinate of the point P.



Sol:

$x_1 = 1 \qquad y_1 = 2$

$x_2 = 5 \qquad y_2 = 3$

$x_3 = 4 \qquad y_3 = 6$

WKT,

$$x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + N_3$$

$$3.3 = (1-4)\, 0.3 + (5-4) N_2 + 4$$

$$\boxed{N_2 = 0.2}$$

$\because N_1 + N_2 + N_3 = 1$

$$0.3 + 0.2 + N_3 = 1$$

$$\boxed{N_3 = 0.5}$$

Then,

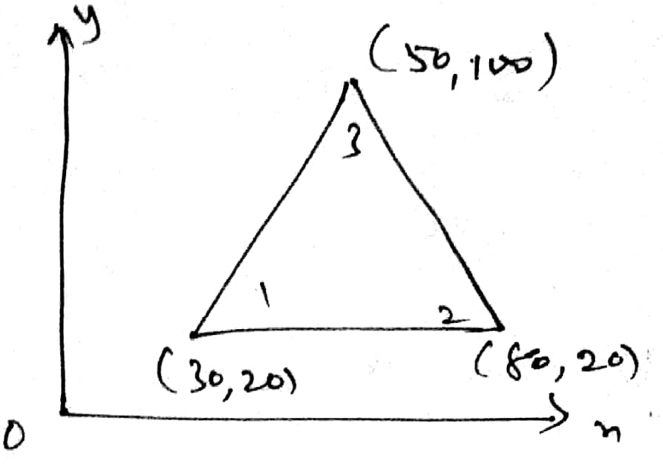
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$y = 0.3(2) + 0.2(3) + 0.5(6)$$

$$\boxed{y = 4.2}$$

② For the plane stress element shown in fig. evaluate the stiffness matrix. Assume $E = 210 \times 10^{3}\ N/mm^2$. $\mu = 0.25$, $t = 10\ mm$.

The Co-ordinates are given in mm.



Sol:

$x_1 = 30 \qquad y_1 = 20$

$x_2 = 80 \qquad y_2 = 20$

$x_3 = 50 \qquad y_3 = 100$.

for CST,

$$[K] = [B]^T [D] [B] A t.$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 30 & 20 \\ 1 & 80 & 20 \\ 1 & 50 & 100 \end{vmatrix}$$

$$A = 2000 \text{ mm}^2$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\beta_1 = y_2 - y_3 = 20 - 100 = -80$$

$$\beta_2 = y_3 - y_1 = 100 - 20 = 80$$

$$\beta_3 = y_1 - y_2 = 20 - 20 = 0$$

$$\gamma_1 = x_3 - x_2 = 50 - 80 = -30$$

$$\gamma_2 = x_1 - x_3 = 30 - 50 = -20$$

$$\gamma_3 = x_2 - x_1 = 80 - 30 = 50$$

$$\therefore [B] = \frac{1}{2 \times 2000}\begin{bmatrix} -80 & 0 & 80 & 0 & 0 & 0 \\ 0 & -30 & 0 & -20 & 0 & 50 \\ -30 & -80 & -20 & 80 & 50 & 0 \end{bmatrix}$$

$$= \frac{1}{400}\begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{40}\begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}$$

for plane stress condition,

$$D = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{210\times10^3}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D] = 56\times10^3\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

Smalls,

$$(k) = \frac{1}{400}\begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix} 56\times10^3\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times$$

$$\frac{1}{400}\begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix} \quad 2000 \times 10.$$

$$K = 7 \times 10^{3} \begin{bmatrix} 269.5 & 60 & -247 & -20 & -22.5 & -40 \\ 60 & 132 & 0 & -72 & -60 & -60 \\ -247 & 0 & 262 & -40 & -15 & 40 \\ -20 & -72 & -40 & 112 & 60 & -40 \\ -22.5 & -60 & -15 & 60 & 37.5 & 0 \\ -40 & -60 & 40 & -40 & 0 & 100 \end{bmatrix}$$

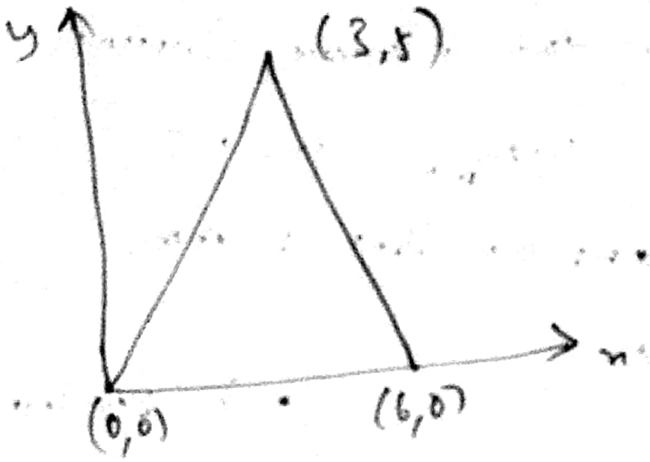
Since the above matrix is Symmetric.

$\therefore$ the Sum of the Values at any Column is 0.

Therefore, Calculated Stiffness matrix is Correct.

(3) Evaluate the element Stiffness matrix for the triangular element as shown in fig under plane strain Condition. Assume,

$E = 200$ Gpa, $\mu = 0.4$ $t = 1$ mm.



Sol:

$A = 15\ mm^2$ $t = 1$

$$[B] = \frac{1}{30}\begin{bmatrix} -5 & 0 & 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 6 \\ -3 & -5 & -3 & 5 & 6 & 0 \end{bmatrix}$$

$$[D] = 8\times10^4\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K = 1.33\times10^3\begin{bmatrix} 84 & 30 & -66 & 0 & -18 & -30 \\ 30 & 52 & 0 & 2 & -30 & -54 \\ -66 & 0 & 84 & -30 & -18 & 30 \\ 0 & 2 & -30 & 52 & 30 & -54 \\ -18 & -30 & -18 & 30 & 36 & 0 \\ -30 & -54 & 30 & -54 & 0 & 108 \end{bmatrix}$$ Ans.

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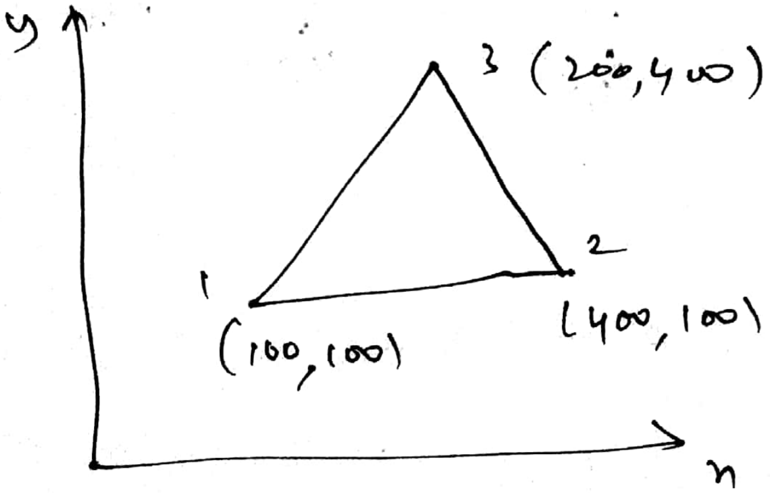
④ For the plane stress element, the nodal displacements are;

$u_1 = 2$ mm $v_1 = 1$ mm

$u_2 = 1$ mm $v_2 = 1.5$ mm

$u_3 = 2.5$ mm $v_3 = 0.5$ mm.

Determine the element stresses. Assume $E = 200$ GN/mm² $\mu = 0.3$ $t = 10$ mm. all dimensions are in mm.



Sol:

$$[\sigma] = [D](e)$$

$$[\sigma] = [D][B][\delta]$$

$x_1 = 100$ $y_1 = 100$

$x_2 = 400$ $y_2 = 100$

$x_3 = 200$ $y_3 = 400$

For plane Stress,

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{2\times 10^5}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D] = \frac{2\times 10^6}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\beta_1 = y_2 - y_3 = 100 - 400 = -300$$

$$\beta_2 = y_3 - y_1 = 400 - 100 = 300$$

$$\beta_3 = y_1 - y_2 = 100 - 100 = 0$$

$$\gamma_1 = x_3 - x_2 = 200 - 400 = -200$$

$$\gamma_2 = x_1 - x_3 = 100 - 200 = -100$$

$$\gamma_3 = x_2 - x_1 = 400 - 100 = 300$$

$$[A] = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 100 & 100 \\ 1 & 400 & 100 \\ 1 & 200 & 400 \end{vmatrix}$$

$$A = 45000 \text{ mm}^2.$$

$$[B] = \frac{1}{2\times 45000}\begin{bmatrix} -300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -200 & 0 & -100 & 0 & 300 \\ -200 & -300 & -100 & 300 & 300 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{900}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix}$$

$$\delta = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1.5 \\ 2.5 \\ 0.5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{2\times 10^6}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}\frac{1}{900}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1.5 \\ 2.5 \\ 0.5 \end{bmatrix}$$

$$= \frac{2\times 10^6}{91\times 900}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{2\times 10^4}{819}\begin{bmatrix} -36 \\ -29 \\ 14 \end{bmatrix} = \begin{bmatrix} -879.12 \\ -708.18 \\ 341.88 \end{bmatrix} N/mm^2.$$

Maximum normal Stress,

$$\sigma_1 = \frac{1}{2}\left[(\sigma_x+\sigma_y) + \sqrt{(\sigma_x-\sigma_y)^2 + 4\tau_{xy}^2} \right]$$

$$= \frac{1}{2}\left[(-879.12 - 708.18) + \sqrt{(-879.12+708.18)^2 + 4(341.88)^2} \right]$$

$$= \frac{1}{2}\left[-1587.3 + 704.81 \right]$$

$$\sigma_1 = -441.24 \ N/mm^2$$

Minimum normal Stress,

$$\sigma_2 = \frac{1}{2}\left[(\sigma_x+\sigma_y) - \sqrt{(\sigma_x-\sigma_y)^2 + 4\tau_{xy}^2} \right]$$

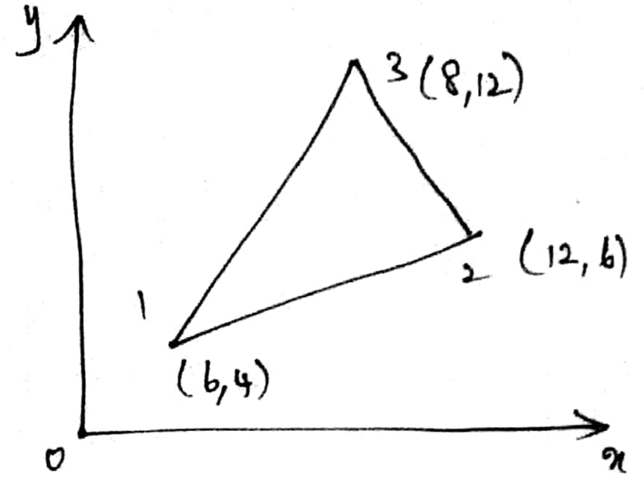
$$= \frac{1}{2}\left[-1587.3 - 704.81 \right]$$

$$\sigma_2 = -1146. \ N/mm^2$$

Principle angle $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\times 341.88}{-879.12+708.18}\right) = \underline{\underline{-37.98^\circ}}$$

5) The element shown in fig is subjected to a temperature change 10°c. Find the load due to temperature change.
Take $E = 200$ Gpa, $\mu = 0.3$, $t = 2$ mm
$\alpha = 7 \times 10^{-6} /°c$. Assume plane stress conditions.
The co-ordinates in mm.



Sol:

$x_1 = 6 \qquad y_1 = 4$

$x_2 = 12 \qquad y_2 = 6$

$x_3 = 8 \qquad y_3 = 12$

$\Delta T = 10°c$.

$\alpha = 7 \times 10^{-6} /°c \qquad \mu = 0.3 \qquad t = 2$ mm.

$E = 200$ Gpa $= 200 \times 10^3$ N/mm^2 $= 2 \times 10^5$ N/mm^2.

$$[F_0] = [B]^T [D] [e_0] A t.$$

↳ Thermal load vector.

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$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 6 & 4 \\ 1 & 12 & 6 \\ 1 & 8 & 12 \end{vmatrix}$$

$$A = 22 \text{ mm}^2.$$

$\beta_1 = y_2 - y_3 = 6 - 12 = -6$

$\beta_2 = y_3 - y_1 = 12 - 4 = 8$

$\beta_3 = y_1 - y_2 = 4 - 6 = -2$

$\gamma_1 = x_3 - x_2 = 8 - 12 = -4$

$\gamma_2 = x_1 - x_3 = 6 - 8 = -2$

$\gamma_3 = x_2 - x_1 = 12 - 6 = 6.$

$$[B] = \frac{1}{44}\begin{bmatrix} -6 & 0 & 8 & 0 & -2 & 0 \\ 0 & -4 & 0 & -2 & 0 & 6 \\ -4 & -6 & -2 & 8 & 6 & -2 \end{bmatrix}$$

$$[B]^T = \frac{1}{44}\begin{bmatrix} -6 & 0 & -4 \\ 0 & -4 & -6 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ -2 & 0 & 6 \\ 0 & 6 & -2 \end{bmatrix}$$

For plane Stress,

$$D = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{2\times10^5}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D] = \frac{2\times10^6}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}$$

$$e_0 = \begin{bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{bmatrix} = \begin{bmatrix} 7\times10^{-5} \\ 7\times10^{-5} \\ 0 \end{bmatrix}$$

$$\therefore \quad [F_0] = [B]^T[D][e_0]\,A\,t.$$

$$\begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix} = \frac{1}{44}\begin{bmatrix} -6 & 0 & -4 \\ 0 & -4 & -6 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ -2 & 0 & 6 \\ 0 & 6 & -2 \end{bmatrix} \frac{2\times10^6}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}\begin{bmatrix} 7\times10^{-5} \\ 7\times10^{-5} \\ 0 \end{bmatrix}(22\times2)$$

$$= \frac{2\times10^{6}\times22\times2}{44\times91}\begin{bmatrix}-60 & -18 & -14\\ -12 & -40 & -21\\ 80 & 24 & -7\\ -6 & -20 & 28\\ -20 & -6 & 21\\ 18 & 60 & -7\end{bmatrix}\begin{bmatrix}7\times10^{-5}\\ 7\times10^{-5}\\ 0\end{bmatrix}$$

$$= \frac{2\times10^{6}}{91}\times7\times10^{-5}\begin{bmatrix}-60 & -18 & -14\\ -12 & -40 & -21\\ 80 & 24 & -7\\ -6 & -20 & 28\\ -20 & -6 & 21\\ 18 & 60 & -7\end{bmatrix}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$= \frac{140}{91}\begin{bmatrix}-78\\ -52\\ 104\\ -26\\ -26\\ 78\end{bmatrix} = \begin{bmatrix}-120\\ -80\\ 160\\ -40\\ -40\\ 120\end{bmatrix} N.$$

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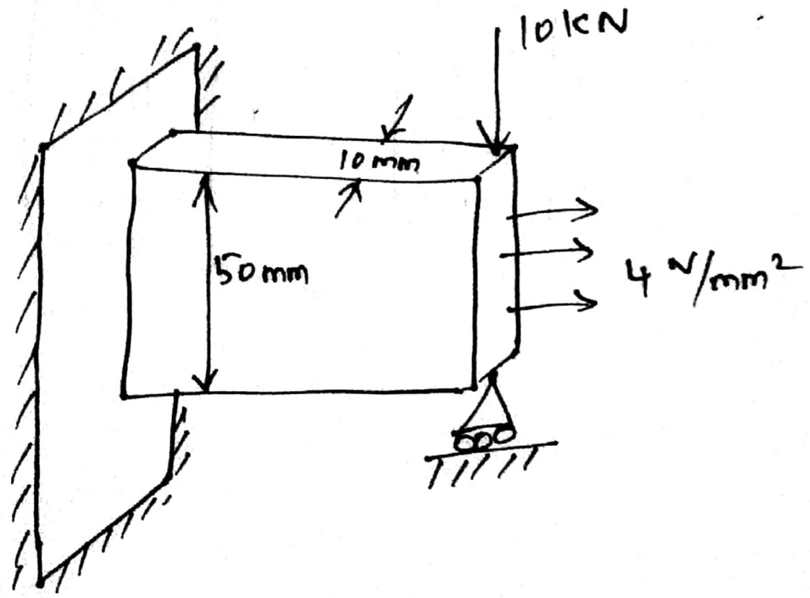
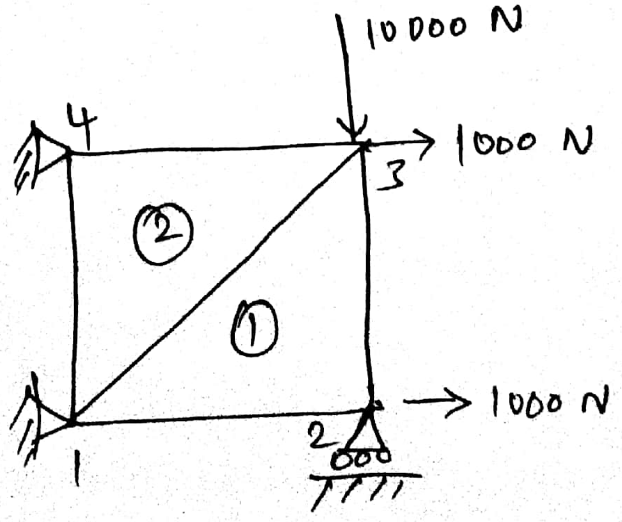
Univeristy Problem :

For a thin plate subjected to the surface traction force along with an external load as shown in fig. determine the nodal displacements and element stresses. Take $t = 10$ mm $E = 210$ GPa $\mu = 0.25$. Analyse the plate using two element model.



Sol:



Nodal Force $= \frac{1}{2} TA$.

$T \rightarrow$ Traction force. (N/mm^2)

$F = \frac{1}{2} \times 4 \times (50 \times 10)$

$F = 1000N$. @ node 2 & 3.