ONE DIMENSIONAL ELEMENTS

$\Rightarrow$

Mohan S R, AP/Mech

UNIT - II

① Consider a bar as shown in fig. An axial load of 200 kN is applied at point P. Take

$A_1 = 2400\ mm^2$ $E_1 = 70 \times 10^9\ N/m^2$, $A_2 = 600\ mm^2$

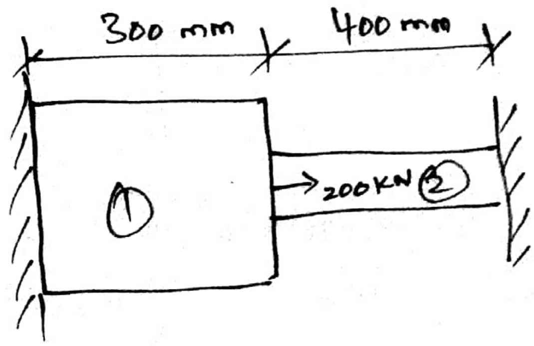
$E_2 = 200 \times 10^9\ N/m^2$.

Calculate the following,

a) The nodal displacement at point P

b) Stress in each material

c) Reaction force.



Apr/May 2017.

Sol:

$A_1 = 2400\ mm^2$

$A_2 = 600\ mm^2$

$l_1 = 300\ mm$

$l_2 = 400\ mm$

$E_1 = 70 \times 10^9\ N/m^2 = 70 \times 10^3\ N/mm^2$

$E_2 = 200 \times 10^9\ N/m^2 = 200 \times 10^3\ N/mm^2$

$P = 200\ kN = 200 \times 10^3\ N.$

$$[F] = [K][\delta]$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element ①

$$K_1 = \frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2400 \times 70 \times 10^3}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_1 = 5.6 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 5.6 & -5.6 \\ -5.6 & 5.6 \end{bmatrix}$$

Element ②

$$K_2 = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{600 \times 200 \times 10^3}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = 3 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$K = K_1 + K_2$$

$$= 10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

Force Vector,

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\times10^5 \\ 0 \end{bmatrix}$$

Displacement Vector,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix}$$

$$[K][u] = [F]$$

$$10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\times10^5 \\ 0 \end{bmatrix}$$

$$\therefore (8.6 u_2) 10^5 = 2\times10^5$$

$$\boxed{u_2 = 0.2325 \text{ mm}}$$

Stresses in element ①,

$$\sigma_1 = E_1 \times \frac{u_2 - u_1}{l_1}$$

$$= 70\times10^3 \left(\frac{0.2325 - 0}{300} \right)$$

$$\sigma_1 = 54.25 \text{ N/mm}^2$$

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Stresses in element ②,

$$\sigma_2 = E_2 \times \frac{u_3 - u_2}{l_2}$$

$$= 200 \times 10^3 \left(\frac{0 - 0.2325}{400} \right)$$

$$\sigma_2 = -116.25 \text{ N/mm}^2$$

Reaction forces,

$$[R] = [K][u] - [F]$$

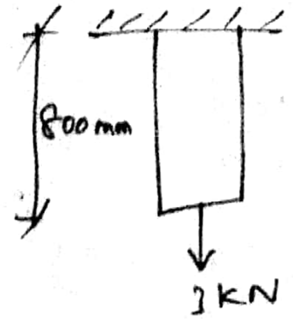
$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0.2325 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2\times10^5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^5 \begin{bmatrix} -1.302 \\ 1.99 \\ -0.697 \end{bmatrix} - \begin{bmatrix} 0 \\ 2\times10^5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} -1.302\times10^5 \\ 0 \\ -0.697\times10^5 \end{bmatrix} \text{ N}$$

(2) A Steel bar of length 800 mm is Subjected to an axial load of 3 kN. Find the nodal displacements of the bar and load Vectors.

Take $E = 2\times10^5$ N/mm². $A = 300$ mm² $\rho = 7800$ kg/m³.



Apr/May 2016.

Sol:

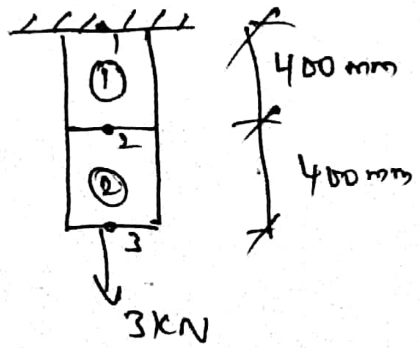
$l = 800$

$F = 3\times10^3$ N

$E = 2\times10^5$ N/mm²

$A = 300$ mm².

$\rho = 7800$ kg/m³ $= 76518$ N/m³ $= 7.65\times10^{-5}$ N/mm³



Force Vector,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \rho A l \begin{bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{bmatrix}$$

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$$= 7.6518\times10^{-5}\times300\times800\begin{bmatrix}1/6\\1/6\\2/3\end{bmatrix}$$

$$\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}=\begin{bmatrix}3.06\\3.06\\12.24\end{bmatrix}$$

Point load 3kN is acting on node 3.

$$\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}=\begin{bmatrix}3.06\\3.06\\12.24+3\times10^3\end{bmatrix}=\begin{bmatrix}3.06\\3.06\\3012.24\end{bmatrix}$$

$$F = k \cdot u$$

$$\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix}=\frac{AE}{3l}\begin{bmatrix}7&1&-8\\1&7&-8\\-8&8&16\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

$$\begin{bmatrix}3.06\\3.06\\3012.24\end{bmatrix}=\frac{2\times10^5\times300}{3\times800}\begin{bmatrix}7&1&-8\\1&7&-8\\-8&8&16\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

$u_1 = 0$ (fixed)

$$\begin{bmatrix}3.06\\3.06\\3012.24\end{bmatrix}=2500\begin{bmatrix}7&1&-8\\1&7&-8\\-8&8&16\end{bmatrix}\begin{bmatrix}0\\u_2\\u_3\end{bmatrix}$$

$$\begin{bmatrix}3.06\\3012.24\end{bmatrix}=2500\begin{bmatrix}7&-8\\-8&16\end{bmatrix}\begin{bmatrix}u_2\\u_3\end{bmatrix}$$

$$2500(7u_2 - 8u_3) = 3.06 \quad \text{——} ①$$

$$2500(-8u_2 + 16u_3) = 3012.24 \quad \text{——} ②$$

Solve,

$$2500(-56u_2 + 112u_3) = 2.108 \times 10^4$$

$$2500(56u_2 - \cancel{64}u_3) = 24.48$$

$$2500(48u_3) = 2.105 \times 10^4$$

$$u_3 = 0.175 \text{ mm.}$$

Sub in ①

$$2500(7u_2 - 8(0.175)) = 3.06$$

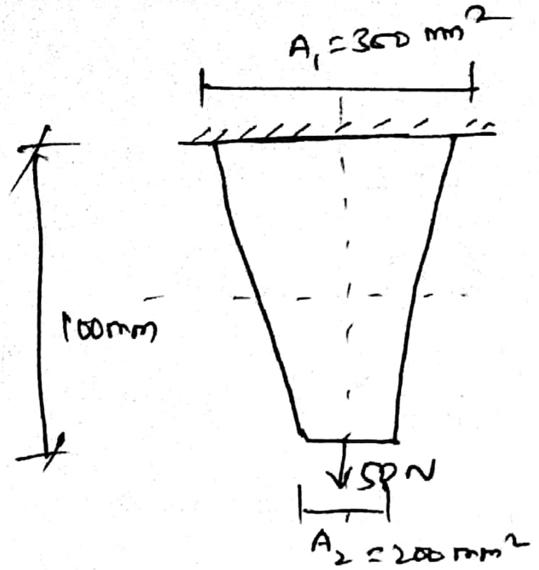
$$17500u_2 - 35000 = 3.06$$

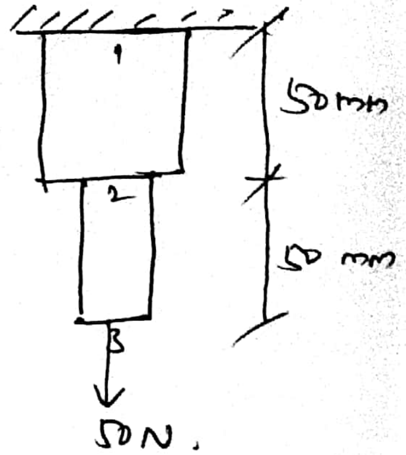
$$u_2 = 2.00 \text{ mm.}$$

③ Using two finite elements, find the Stress distribution in a uniformly tapering bar a circular cross sectional area $3\,cm^2$ ($300\,mm^2$) & $2\,cm^2$ ($200\,mm^2$) at their ends, length 100 mm, subjected to an axial tensile load of 50 N at smaller end and fixed at larger end. Take the value of Young's modulus $2 \times 10^5\ N/mm^2$.

Nov/Dec 2017

Sol:





at node 1, $A_1 = 3\,cm^2 = 300\,mm^2$

at node 3, $A_3 = 2\,cm^2 = 200\,mm^2$

Area at node 2,

$$A_2 = \frac{A_1 + A_3}{2} = \frac{3+2}{2}$$

$$A_2 = 2.5\,cm^2.$$

Avg. Area for ① $\Big\}$, $A_1 = \frac{A_1 + A_2}{2} = \frac{3+2.5}{2}$

$$A_1 = 2.75\,cm^2 = 2.75 \times 10^2\,mm^2.$$

Avg. Area for ② $\Big\}$ $A_2 = \frac{A_2 + A_3}{2} = \frac{2.5+2}{2}$

$$A_2 = 2.25\,cm^2 = 2.25 \times 10^2\,mm^2.$$

Tensile load at node 3, = 50N.

$E = 2 \times 10^5\,N/mm^2.$

for element ①,

$$K_1 = \frac{A_1 E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{275 \times 2 \times 10^5}{50}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_1 = 10^5\begin{bmatrix} 11 & -11 \\ -11 & 11 \end{bmatrix}$$

for element ②,

$$K_2 = \frac{A_2 E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{225 \times 2 \times 10^5}{50}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = 10^5\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

Global Stiffness Matrix,

$$K = K_1 + K_2$$

$$[K] = 10^5\begin{bmatrix} 11 & -11 & 0 \\ -11 & 20 & -9 \\ 0 & -9 & 9 \end{bmatrix}$$

Applying boundary Conditions,

$$u_1 = 0.$$

Neglecting rows & Columns,

$$10^5 \begin{bmatrix} 20 & -9 \\ -9 & 9 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

$$10^5 [20u_2 - 9u_3] = 0 \quad \text{——} (1)$$

$$105 [-9u_2 + 9u_3] = 50 \quad \text{——} (2)$$

solve,

$$105 [11 u_2] = 50$$

$$\boxed{u_2 = 4.5 \times 10^{-5} \text{ mm}}.$$

$$10^5 [20 (4.5 \times 10^{-5}) - 9u_3] = 0.$$

$$\boxed{u_3 = 1.01 \times 10^{-4} \text{mm}.}$$

<u>Stresses :</u>

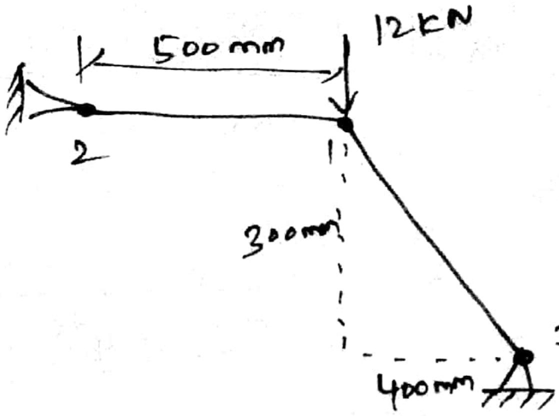
$$\sigma_1 = \frac{E (u_2 - u_1)}{l_1} = \frac{2 \times 105 (4.545 \times 10^{-5} - 0)}{50}$$

$$\sigma_1 = 0.1816 \text{ N/mm}^2$$

$$\sigma_2 = \frac{E (u_3 - u_2)}{l_2} = \frac{2 \times 10^5 (1.01 \times 10^{-4} - 4.54 \times 10^{-5})}{50}$$

$$\sigma_2 = 0.222 \text{ N/mm}^2.$$

(4) For the two bar truss shown in fig. determine the displacements of node 1 and the stress in the element 1-3.

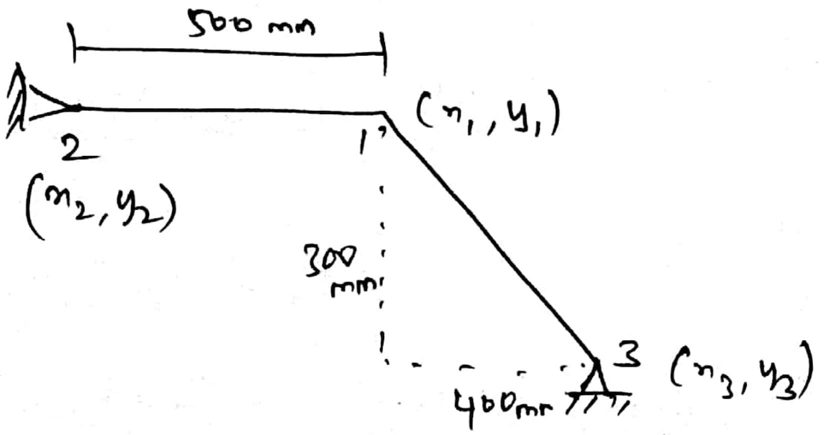


E = 70 GPa

A = 200 mm²

Nov/Dec 2018

Sol:



$$E = 70\ GPa = 70 \times 10^9\ N/m^2$$

$$E = 70 \times 10^3\ N/mm^2$$

$$A = 200\ mm^2$$

For element (1)

$$l_{e_1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-500-0)^2 + (0-0)^2}$$

$$l_{e_1} = 500\ mm$$

Direction of Cosines $l_1 = \frac{x_2 - x_1}{l_{e_1}}$

$$= \frac{-500 - 0}{500}$$

$$l_1 = -1$$

$$m_1 = \frac{y_2 - y_1}{l_{e_1}} = \frac{0-0}{500} = 0.$$

For element ②,

$$l_{e_2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$= \sqrt{(400-0)^2 + (-300-0)^2}$$

$$l_{e_2} = 500 \text{ mm}.$$

Direction of Cosines, $l_2 = \frac{x_3 - x_1}{l_{e_2}} = \frac{400}{500}$

$$l_2 = 0.8$$

$$m_2 = \frac{y_3 - y_1}{l_{e_2}} = \frac{-300 - 0}{500} = -0.6.$$

$$[K]_1 = \frac{A_1 E_1}{l_{e_1}} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & -m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & -l_1 m_1 \\ -l_1 m_1 & -m_1^2 & -l_1 m_1 & -m_1^2 \end{bmatrix}$$

$$[k]_1 = \frac{2000 \times 70 \times 10^3}{500} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k_1] = 28 \times 10^3 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k_2] = \frac{200 \times 70 \times 10^3}{500} \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$[k_2] = 28 \times 10^3 \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$[k] = [k_1] + [k_2]$$

$$[k] = 28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ -0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix}$$

$$[k][u] = [F]$$

$$28\times10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ -0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

$$F_2 = -12\times10^3\,N$$

$$u_3 = u_4 = u_5 = u_6 = 0.$$

$$28\times10^3 \begin{bmatrix} 1.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -12\times10^3 \end{bmatrix}$$

$$28\times10^3 (1.64u_1 - 0.48u_2) = 0. \quad \text{—— (1)}$$

$$28\times10^3 (-0.48u_1 + 0.36u_2) = -12\times10^3 \quad \text{—— (2)}$$

by Solve,

$$u_2 = -1.952 \text{ mm}$$

$$u_1 = -0.571 \text{ mm}.$$

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Stresses, 1-2,

$$\sigma = \frac{E}{l_{e_1}} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= \frac{70\times10^3}{500} \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -0.571 \\ -1.952 \\ 0 \\ 0 \end{bmatrix}$$

$\sigma = 79.94 \; N/mm^2$ (Tensile load).

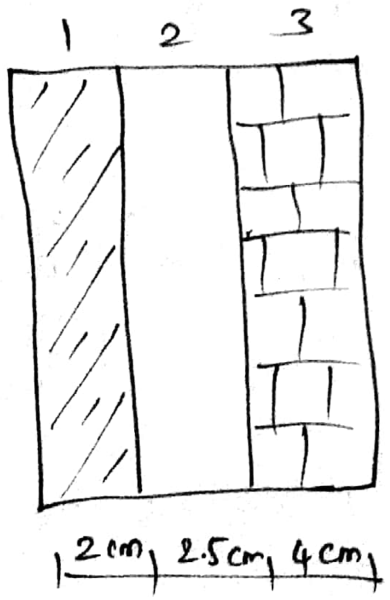
Stresses 1-3,

$$\sigma = \frac{E}{l_{e_2}} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_6 \end{bmatrix}$$

$$= \frac{70\times10^3}{500} \begin{bmatrix} -0.8 & 0.6 & 0.8 & -0.6 \end{bmatrix} \begin{bmatrix} -0.571 \\ -1.952 \\ 0 \\ 0 \end{bmatrix}$$

$\sigma = -100 \; N/mm^2$. (compressive load)

⑤ A Composite Wall consists of three materials as shown in fig. The inside wall temperature is 100°c and the outside air temperature is 50°c with a Convection Co-efficient h of 10 W/cm²°c. Determine the temperature along the Composite wall if the Conduction Co-efficients of the three materials are as follows in the fig.



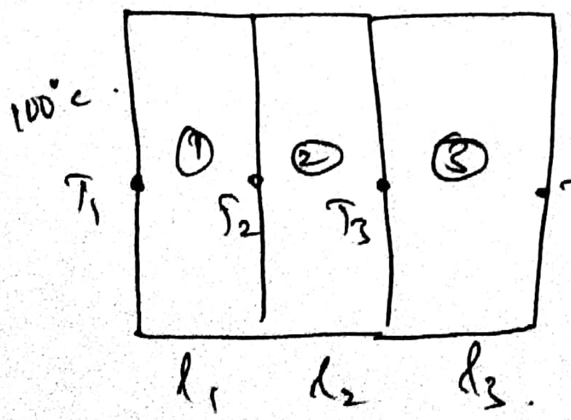
$k_1 = 70\ W/m°K$

$k_2 = 40\ W/m°K$

$k_3 = 20\ W/m°K$

Nov/Dec 2019.

Sol:



$T_\infty = 50°c$

$h = 10\ W/cm^2°c$

$l_1 = 2\,cm \quad l_2 = 2.5\,cm \quad l_3 = 4\,cm.$

element ① :

$$[k_1] = \frac{k_1 A_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$A = 1\,cm^2.$

$$= \frac{70\times 1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 35\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_1] = \begin{bmatrix} 35 & -35 \\ -35 & 35 \end{bmatrix}$$

element ②

$$[k_2] = \frac{k_2 A_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{40\times 1}{2.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_2] = \begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix}$$

element ③

$$[k_3] = \frac{20\times 1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(due to Convection)

$$[k_3] = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 15 \end{bmatrix}$$

FEA, eqn,

$$[K][T] = [F]$$

$$F_1 = F_2 = F_3 = 0,$$

$$F_4 = hAT_\infty = 10 \times 1 \times 323 = 3230\,W.$$

$$T_1 = 100^\circ c = 373\,K.$$

global Matrix,

$$\begin{bmatrix} 35 & -35 & 0 & 0 \\ -35 & (35+16) & -16 & 0 \\ 0 & -16 & 16+5 & -5 \\ 0 & 0 & -5 & 15 \end{bmatrix} \begin{bmatrix} 373 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3230 \end{bmatrix}$$

$$-13055 + 51\,T_2 - 16\,T_3 = 0 \quad \text{——— } 1$$

$$-16\,T_2 + 21\,T_3 - 5\,T_4 = 0 \quad \text{——— } 2$$

$$-5\,T_3 + 15\,T_4 = 3230 \quad \text{——— } 3$$

$$5T_3 = -3230 + 15T_4$$

$$T_3 = 3T_4 - 646 \quad \text{——— } 4$$

Sub in 2,

$$-16T_2 + 21(3T_4 - 646) - 5T_4 = 0.$$

$$-16T_2 + 58T_4 - 13566 = 0.$$

$$T_2 = 3.62T_4 - 847.8 \quad \text{——— } 5$$

Sub ④ in ⑦

$$-13055 + 51 T_2 - 16 (3T_4 - 646) = 0$$

$$-13055 + 51 T_2 - 48 T_4 + 10336 = 0$$

$$51 T_2 = 2719 + 48 T_4.$$

$$T_2 = 53.31 + 0.94 T_4. \quad — ⑥$$

Equate eqn ⑤ & ⑥

$$3.62 T_4 - 847.8 = 53.31 + 0.94 T_4.$$

$$2.68 T_4 = 901.11$$

$$\boxed{T_4 = 336.2 K}$$

∴

$$T_2 = 53.31 + 0.94 (336.2)$$

$$\boxed{T_2 = 369.33 K}$$

∴ $$T_3 = 3(336.2) - 646$$

$$\boxed{T_3 = 362.6 K}$$