WEIGHTED RESIDUAL METHODS

RITZ TECHNIQUE ⟹

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UNIT-1

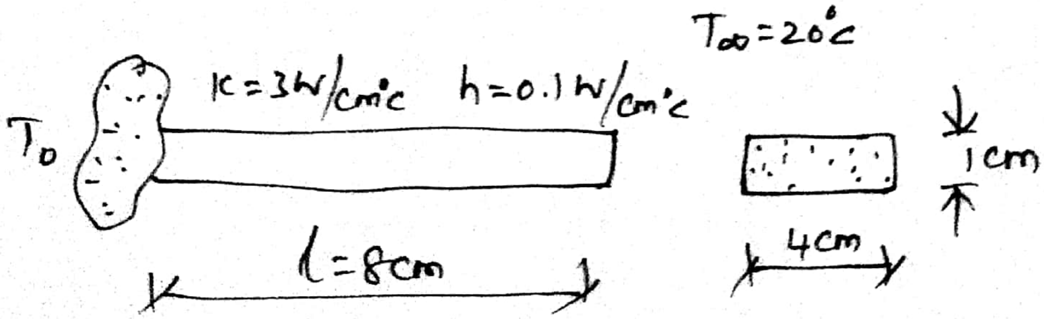
Functional Approximation Method

* Weighted Residual Method
* Variational Method (Rayleigh Ritz)

① The governing equation for one dimensional heat transfer through a fin of length l attached to a hot source as shown in fig. is given by

$$\frac{d}{dx}\left[-kA\frac{dT}{dx}\right] + hP\,(T-T_\infty) = 0 .$$

If the free end of the fin is insulated, give the boundary conditions and determine using the collocation technique the temperature distribution in the fin. Report the temperature at the free end.



Apr/May 2018 & 2019.

Sol:

$$\frac{d}{dx}\left[-kA\frac{dT}{dx}\right] + hp(T - T_\infty) = 0$$

$L = 8\ cm$, $t = 1\ cm$ $b = 4\ cm$ $k = 3\ W/cm°C$

$h = 0.1\ W/cm^2 °C$ $T_\infty = 20°C = 293K$

$T_1 = 200°C = 473K.$

Perimeter of the fin $= 2(b+t)$

$= 2(4+1) = 10\ cm.$

Area $= b \times t = 4\ cm^2.$

Assume 2^{nd} order polynomial eqn,

$$T = a_0 + a_1 x + a_2 x^2 \quad \text{——} (1)$$

at $x = 0$, $T = 437K$

$$a_0 = 473$$

$$\therefore T = 473 + a_1 x + a_2 x^2 \quad \text{——} (2)$$

at $x = 8$, $H.T = 0$ or $\frac{dT}{dx} = 0.$

$$\frac{dT}{dx} = a_1 + 2a_2 x = 0$$

$$a_1 = -16a_2$$

$$T = 473 - 16a_2x + a_2x^2$$

$$T = 473 + (x^2 - 16x)a_2 . \quad \text{—} (3)$$

from above eqn,

$$\frac{dT}{dx} = (2x - 16)\, a_2$$

$$\frac{d^2T}{dx^2} = 2a_2$$

Substitute the value of $\frac{d^2T}{dx^2}$ in governing eqn to find residual R,

$$R = (x^2 - 16x - 24)\, a_2 + 180$$

Solving by point collocation,

$$R = 0$$

$$(x^2 - 16x - 24)\, a_2 + 180 = 0 \qquad x = l/2$$

$$x = 4\,cm.$$

$$(16 - 64 - 24)\, a_2 = -180$$

$$-72a_2 = -180$$

$$a_2 = 2.5$$

Sub in trial function,

$$T = 473 + (x^2 - 16x)\, 2.5$$

Temperature at free end $x = L = 8$,

$$T = 473 + (8^2 - 16(8))\, 2.5 = 313K = 40^\circ c.$$

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(2) A physical phenomenon is governed by the differential equation $\frac{d^2w}{dx^2} - 10x^2 = 5$ for $0 \le x \le 1$. The boundry conditions are given by $w(0) = w(1) = 0$. Assuming a trial solution $w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. determine using Galerkin method the variation of 'w' with respect to x.

Nov/Dec 2016.

Sol:

$$\frac{d^2w}{dx^2} - 10x^2 = 5 \qquad 0 \le x \le 1$$

boundry conditions are $w(0) = 0$, $w(1) = 0$

Assume trial function,

$$W(x) = a_1 x (x - 1)$$

at $x = 0$, $w = 0$

$x = 1$, $w = 0$

Hence it satisfies the boundary conditions.

$$w = a_1 x(x-1) = a_1(x^2-x)$$

$$dw/dx = a_1(2x-1)$$

$$\frac{d^2w}{dx} = 2a_1$$

$$\therefore R = 2a_1 - 10x^2 - 5$$

The trial function $x(x-1)$ is considered as weighting function,

$$F_1(x) = x(x-1)$$

Using galerkin method,

$$\int_0^1 F_1(x) \cdot R \cdot dx = 0$$

$$\int_0^1 x(x-1)(2a_1 - 10x^2 - 5)\,dx = 0$$

$$\int_0^1 (x^2-x)(2a_1 - 10x^2 - 5)\,dx = 0$$

$$\int_0^1 (2a_1x^2 - 10x^4 - 5x^2 - 2a_1x + 10x^3 + 5x)\,dx = 0$$

$$2a_1\left(\frac{x^3}{3}\right)_0^1 - 10\left(\frac{x^5}{5}\right)_0^1 - 5\left(\frac{x^3}{3}\right)_0^1 - 2a_1\left(\frac{x^2}{2}\right)_0^1 + 10\left(\frac{x^4}{4}\right)_0^1 + 5\left(\frac{x^2}{2}\right)_0^1 = 0$$

$$\frac{2a_1}{3} - 2 - \frac{5}{3} - a_1 + 5/2 + 5/2 = 0.$$

$$\frac{2a_1}{3} - a_1 = -\frac{4}{3}$$

$$-\frac{a_1}{3} = \frac{-4}{3}$$

$$\boxed{a_1 = 4}$$

Hence,

$$w = ax(x-1)$$
$$= 4x(x-1)$$
$$w = 4x^2 - 4x.$$

③ The following differential equation is available for a physical phenomenon.

$$\frac{d^2y}{dx^2} + 50 = 0 \quad ; \quad 0 \leq x \leq 10$$

The trial function is $y = a_1 x(10-x)$

The boundary Conditions are $y(0) = 0$ $y(10) = 0$.

Find the Value of the parameter a_1 by the following methods.

i) Least Square

ii) Galerkin's

Sol:

$$\frac{d^2y}{dx^2} + 50 = 0 \qquad y(0) = 0$$
$$y(10) = 0$$

Trial fn,

$$y = a_1 x(10 - x)$$

At $x = 0, \quad y = 0$

$x = 10, \quad y = 0.$

it satisfies the boundary conditions.

To find Residual,

$$y = a_1 x(10 - x)$$
$$y = 10 x a_1 - x^2 a_1$$
$$\frac{dy}{dx} = 10a_1 - 2xa_1$$
$$\frac{d^2y}{dx^2} = -2a_1$$

$\therefore \quad R = -2a_1 + 50.$

i) Least Square method

$$\frac{\partial S}{\partial a_1} = 0.$$

$$S = \int R^2 \cdot dx.$$

$$\frac{\partial S}{\partial a_1} = \int \frac{\partial R^2}{\partial a_1}\, dx = 0$$

$$\Rightarrow \int R \cdot \frac{\partial R}{\partial a_1}\, dx = 0$$

$$\int_0^{10} (-2a_1+50)(-2)\, dx = 0$$

$$\int_0^{10} (4a_1 - 100)\, dx = 0$$

$$(4a_1 - 100)(x)_0^{10} = 0$$

$$(4a_1 - 100)\, 10 = 0$$

$$4a_1 = 100$$

$$a_1 = 25$$

$\therefore$ trial fn, $y = -25\, x(10-x)$.

ii) <u>Galerkin's</u>

$$\int_0^{10} f_1(x) \cdot R\, dx = 0$$

$$\int_0^{10} a_1 x(10-x) \cdot (-2a_1+50)\, dx = 0$$

$$\int_0^{10} (10a_1 x - a_1 x^2)(-2a_1 + 50)\, dx = 0$$

$$\int_0^{10} \left(-20a_1^2 x + 2a_1^2 x^2 + 500a_1 x - 50a_1 x^2\right) dx = 0$$

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$$\left[-20a_1^2\frac{x^2}{2}+2a_1^2\frac{x^3}{3}+500a_1\frac{x^2}{2}-50a_1\frac{x^3}{3}\right]_0^{10}=0$$

$$-20a_1^2\left(\frac{100}{2}\right)+2a_1^2\left(\frac{1000}{3}\right)+500a_1\left(\frac{100}{2}\right)-50a_1\left(\frac{1000}{3}\right)=0$$

$$-1000a_1^2+666.66a_1^2+25000a_1-16666.66a_1=0$$

$\div a_1$,

$$-1000a_1+666.66a_1+25000-16666.66=0$$

$$-333.34a_1=-8333.34$$

$$a_1=24.99$$

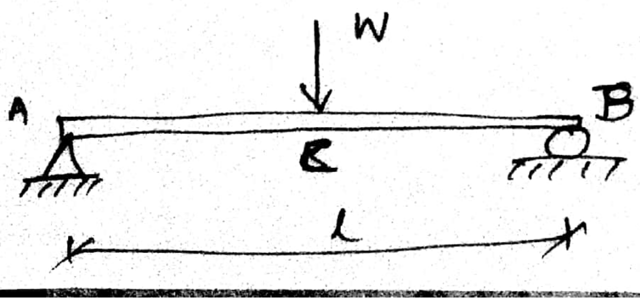
$\therefore$ trial function,

$$y=a_1x(10-x)$$

$$y=24.99\,x(10-x)$$

$$y\approx 25x(10-x).$$

(4) A beam AB of span 'l' simply supported at ends and carrying a concentrated load W at the centre 'C' as shown in fig. Determine the deflection at midspan by using Rayleigh-Ritz Method and compare with exact solution.



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Sol:

The total potential energy,

$$\Pi = U - W.$$

$U \to$ Strain Energy

$W \to$ work done

Strain energy for beam,

$$U = \frac{EI}{2} \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx.$$

$y \to$ deflection

$E \to$ Young's modulus

$I \to$ area M.I.

polynomial eqn,

$$y = a_1 + a_2 x + a_3 x^2 \quad (3 \text{ term})$$

at $x = 0$ $y = 0$

$x = l$ $y = 0$

$$\boxed{0 = a_1}$$

$$0 = a_1 + a_2 l + a_3 l^2$$

$$a_2 l + a_3 l^2 = 0$$

$$\boxed{a_2 = -a_3 l.}$$

$\therefore \quad y = -a_3 l x + a_3 x^2$

$$y = a_3 (x^2 - l x).$$

$$dy/dx = a_3 (2x - l)$$

$$d^2y/dx^2 = 2a_3.$$

$$\therefore \quad U = \frac{EI}{2} \int_0^l (2a_3)^2 \, dx$$

$$= \frac{EI}{2} 4 a_3^2 l = 2EI a_3^2 l$$

Work done $W = \cancel{W} \cdot y_{max}$.

y_{max} @ $x = l/2$.

$$= W \left[a_3 (x^2 - l x) \right]_{x = l/2}$$

$$= W a_3 \left[(l/2)^2 - l (l/2) \right]$$

$$= W a_3 \left[l^2/4 - l^2/2 \right]$$

$$W = -W a_3 l^2 / 4.$$

Total P.E,

$$\Pi = U - W$$

$$\Pi = 2EI a_3^2 l - \left(- W a_3 l^2/4 \right)$$

$$\Pi = 2EI a_3^2 l + W a_3 \frac{l^2}{4}$$

for minimum P.E,

$$\frac{\partial \pi}{\partial a_3} = 0$$

$$4EI a_3 l = -\frac{\cancel{W}l^2}{4}.$$

$$a_3 = -\frac{Wl^2}{4} \times \frac{1}{4EIl}$$

$$a_3 = -\frac{Wl}{16EI}.$$

$$\therefore y = a_3(x^2 - lx)$$

$$y = -\frac{Wl}{16EI}(x^2 - lx)$$

for max. deflection, $x = l/2$,

$$y_{max} = -\frac{Wl}{16EI}\left(\frac{l^2}{4} - \frac{l^2}{2}\right)$$

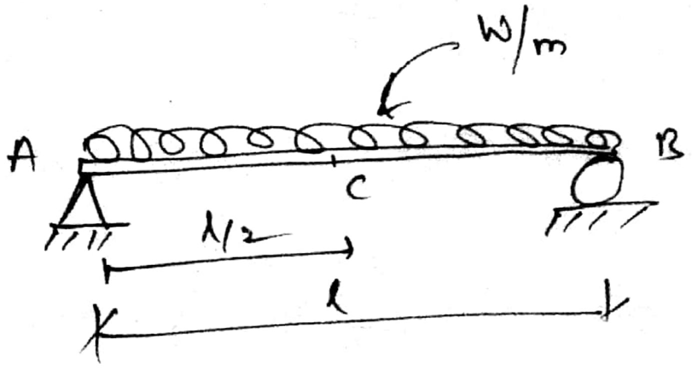
$$y_{max} = \cancel{}\ \frac{Wl^3}{64EI}$$

↑ approxiamate solution.

<u>Exact Solution</u> :

$$y_{max} = \frac{Pl^3}{48EI} \approx \frac{Wl^3}{48EI}.$$

⑤ Find the deflection at the centre of the Simply Supported beam of Span length 'l' Subjected to uniformly distributed load throughout its length as shown in fig. using i) Point Collocation ii) Subdomain method.



Apr/May 2017.

Sol:

governing eqn, for UDL,

$$EI \frac{d^4y}{dx^4} - w = 0. \qquad 0 \leq x \leq l.$$

boundary Conditions are Satisfies the eqns.

@ $x = 0$ $\quad y = 0$

$x = l$ $\quad y = 0$.

Let trial function,

$$y = a \sin \frac{\pi x}{l}.$$

$$\frac{dy}{dx} = a\frac{\pi}{l}\cos\frac{\pi x}{l}.$$

$$\frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}.$$

$$\frac{d^3y}{dx^3} = -a\frac{\pi^3}{l^3}\cos\frac{\pi x}{l}.$$

$$\frac{d^4y}{dx^4} = a\frac{\pi^4}{l^4}\sin\frac{\pi x}{l}.$$

from governing eqn,

$$R = EI\,a\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} - w.$$

i) Point Collocation :

$$R = 0.$$

$$EI\,a\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} - w = 0$$

$$EI\,a\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} = w.$$

to get Max. deflection, $x = l/2$.

$$EI\,a\frac{\pi^4}{l^4}\sin\frac{\pi.(l/2)}{l} = w$$

$$EI\,a\frac{\pi^4}{l^4} = w$$

$$a = \frac{w\,l^4}{\pi^4.EI}$$

$\therefore$ Trial fn,
$$y = \frac{w l^4}{\pi^4 EI} \sin \frac{\pi x}{l}.$$

At $x = l/2$,
$$y_{max} = \frac{w l^4}{\pi^4 EI} \sin \frac{\pi (l/2)}{l}.$$

$$y_{max} = \frac{w l^4}{\pi^4 . EI} = \frac{w l^4}{97.4 EI}$$

ii) <u>Sub domain</u> :

$$\int_0^l R \, dx = 0.$$

$$\int_0^l \left(EI a \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} - w \right) dx = 0$$

$$\left[a EI \frac{\pi^4}{l^4} \left(- \cos \frac{\pi x}{l} \right) \left(l/\pi \right) - wx \right]_0^l = 0$$

$$- a EI \frac{\pi^3}{l^3} \left(\cos \pi - \cos 0 \right) - w l = 0$$

$$- a EI \frac{\pi^3}{l^3} (-1 - 1) = w l.$$

$$a = \frac{w l^4}{2 \pi^3 EI} = \frac{w l^4}{62 EI}.$$

$\therefore$

$$y = \frac{w l^4}{62 EI} \sin \pi x / l \qquad @ \, x = l/2.$$

$$y_{max} = \frac{w l^4}{62 EI} \sin \pi/2 = \frac{w l^4}{62 EI} \quad //.$$